The following brief notes were presented:

(7) "A device for the trisection of an angle." Professor Christian Hornung, Heidelberg University.

(8) "The degree of convergence of a Fourier series of a limited number of terms." Professor T. M. Focke, Case School of Applied Science.

(9) "The evaluation of a certain volume integral." Professor F. Anderegg, Oberlin College.

(10) "Some cases of the cubic equation." Professor William Hoover, Ohio University.

(11) "Descriptive geometry analyses for common geometry constructions." Professor G. N. Armstrong, Ohio Wesleyan University.

(12) "An interesting combination of figures." Professor F. B. Wiley, Denison University.

(Note: Owing to lack of time (11) and (12) were omitted.)

Following are abstracts of the papers presented.

(1) In considering the question as to what elective courses should follow the calculus in the ordinary college, Dr. A. E. Young suggested that all such courses could be classified as algebraic, geometric, functional, or applied mathematics. He favored courses taken from the first two classifications for the average student, a functional course for the exceptionally brilliant, and courses in applied mathematics for the prospective engineer.

In discussing this paper, Professor Morris called attention to the fact that the ordinary freshman course covers college algebra, trigonometry, and analytical geometry. These courses are so fundamental and are done in such a superficial manner that he advocates following the calculus with a thorough course in plane and solid analytical geometry in the junior year, followed in the senior year with a course in the theory of equations and the elements of the theory of numbers. A student with this preparation is prepared to proceed in any one of many directions.

In making up a twelve-hour program beyond the calculus, Professor Rasor would include a course in algebra, one in geometry, say plane and solid analytics; and then without question a course in differential equations, or advanced calculus inclusive of differential equations—something to give contact with the practical and usable.

In his discussion, Professor Graber noted that trigonometry and analytical geometry are generally presented by the text book and problem method, but in the calculus freer and more general methods of instruction may be introduced, leading up to the lecture plan. Most of our colleges offer at least one advanced course in each of the four divisions, analysis, mechanics, algebra, and geometry, without special attempt to correlate these subjects with one another or with science and engineering. There should be a well-defined connectivity in the post-calculus courses both with the different fields of mathematics and the applications of science. A spiral extension of elementary mathematics into the outlying

fields of number theory, modern geometry, substitution groups, mechanics, and the fundamental notions of modern mathematics in general, affords abundant material for lectures in the courses following the calculus.

(2) Professor Allen gave a few of the fundamental theorems on hypercomplex number systems, with enough of the proofs to indicate their elementary character; and by use of these showed the only real number systems in which division is unambiguous to be the real system, the ordinary complex system, and the real quaternion system.

(3) In Professor Moore's paper, a brief account of the history of divergent series was given. Some of the principal methods of summation were described, and a few of their more important applications were indicated.

(4) Professor Yanney showed that it is possible, as the laws in Ohio are administered, for a college graduate to qualify for a provisional state certificate to teach in the high schools without any special scholastic training in any subject. This is at variance with what experience has shown to be the best preparation for efficient teaching in secondary schools. However, in spite of this deficiency in the laws, there is good evidence that the better colleges, pretending to prepare secondary teachers, demand fully two years in college mathematics including calculus, followed by special courses in the history and teaching of mathematics. The present situation offers the new Association an opportunity to use its influence in the direction of a better public sentiment in the matter of scholastic equipment of the secondary teacher of mathematics.

(5) Professor Glazier insisted that the college must consider the needs of the secondary teachers in mathematics. What they teach, and how they teach it, will determine largely the place of mathematics in the high school curriculum. In addition to the regular undergraduate courses in mathematics the college should also give: (1) A survey course dealing with the fundamental concepts of algebra and geometry, and relating the college mathematics to the secondary. (2) A pedagogical course for consideration of questions definitely concerned with the teaching of mathematics. (3) A history course which should give the historical background of the subject and familiarity with the mathematical literature and mathematical activities.

(6) According to Professor Faught, the minimum academic requirements for unqualified endorsement for a teacher of high-school mathematics should be (1) a four-year high school course with three years of mathematics and one year of physics. (2) A college course with two years of mathematics, including college algebra, trigonometry, analytical geometry, and calculus, and one year of physics. The professional requirements should be (1) one year of educational psychology, principles of teaching and history of education. (2) One year of special methods in algebra and geometry, with the history of mathematics. (3) One half year of observation and practice teaching of algebra and geometry.

(7) Professor Hornung described a piece of mechanism invented by one of his pupils, about eighteen years of age, whereby any angle may be trisected. It consists of a pencil, a string, and a right triangular prism of small height, so

connected that when the base of the prism is moved along the bisector of the angle, as directrix, the pencil will describe a curve, which, by a proper selection of starting point and focus, will pass through a trisection point of any circular arc subtending the angle whose center is at the vertex of the angle. The curve so drawn is an hyperbola.

(8) It can be shown, theoretically, that the r terms of a series of the usual Fourier form that will best represent all the points on a given curve between 0 and 2π, will be the first r terms of the infinite Fourier series. Professor Focke showed, by means of machine drawn curves, that it is possible to find curves whose equations are of the same form and number of terms, which will give a much better approximation to the given curve over a very considerable part of the interval.

(9) On encountering the problem of finding the volume of the solid bounded by the surface $(x^2/a^2 + y^2/b^2 + z^2/c^2)^2 = x^2/a^2 + y^2/b^2 - z^2/c^2$, it occurred to Professor Anderegg that the solution could be simplified by using an auxiliary solid whose surface is obtained from the given surface by putting $x/a = x'$, $y/b = y'$, $z/c = z'$, so that

$$V = \iiint dxdydz = abc \iiint dx'dy'dz' = abcV'.$$

The auxiliary surface $(x'^2 + y'^2 + z'^2)^2 = x'^2 + y'^2 - z'^2$ is obtained by revolving the lemniscate $(x^2 + z^2)^2 = x^2 - z^2$, or $\rho^2 = \cos 2\theta$ about the Z-axis, and the volume bounded by it is easily found. The solution of many problems in quadrature and cubature can be simplified by this device.

(10) The instances of cubic equations presented by Professor Hoover consisted mainly of a number of interesting cases of the cubic resulting from discussions in the applications of pure mathematics over a considerable range of subjects, giving concreteness to the instances adduced.

In his address before the general session on Saturday morning, Professor Pitcher called attention to the responsibility of a given college department to those students who can take but little work in the department but who would acquire as broad a knowledge of the subject as possible. The problem of discharging this responsibility is a very difficult one for mathematics on account of its technical character but very important on account of its many values, especially on account of its central position in scientific learning. Probably an improvement of current conditions may be secured by offering a modified first course to be taken by all alike. Such a course could contain a somewhat broader selection of material than at present and omit certain technique without interfering with the values usually attached to a first course. A further help would be an historical synoptic course open to juniors and seniors who by their first two years of college work have shown themselves capable of mature thought and serious work.

G. N. ARMSTRONG,
Secretary.

PASCAL LINE EQUATIONS AND SOME CONSEQUENCES.

By R. D. BOHANNAN, Ohio State University.

I know of no attempt to give the equations of the Pascal lines of the inscribed hexagon of a conic except that made by Salmon in Articles 267–8 of his *Conic Sections*, and this does not reach any definite form.

Salmon takes *abcdef* as the hexagon order; $ab = 0$, as the equation of ab (*specifying no particular form of equation*).

The conic circumscribing *abcd*, *defa* is either

$$ab \cdot cd = ad \cdot bc \tag{1}$$

or,

$$de \cdot fa = ad \cdot ef; \tag{2}$$

$$\therefore \quad ab \cdot cd - de \cdot fa = ad(bc - ef). \tag{3}$$

The left hand side of (3) (a conic about $ab - de - cd - af$) thus splits into two factors; one is the diagonal ad; therefore the other, $bc - ef = 0$, is the other diagonal, or the Pascal line, since (3) shows that the points ab, de; cd, fa; bc, ef, are on this line.

All that this shows is that the equation of the Pascal may be expressed in terms of those of a pair of opposite sides, and that for any particular hexagon and conic it could be gotten by an appropriate process of subtraction and factoring.

Clearly, Salmon's result (the difference of the equations of a pair of opposite sides) would not, in general, hold if the equations were in "normal form."

Moreover if Salmon's process in Art. 268, to prove concurrency of Pascal lines in a Steiner's g-point, is applied to a Kirkman's h-point, *we may have exactly the same sets of equations to prove concurrency for two entirely different sets of lines* (one line in common).

Consider the g-point and h-point:[1]

[1] This notation is different from Salmon's and better. Salmon indicates the Pascal lines by pairs of horizontal lines, $\left(\frac{ab}{ed}\right)$ indicating the point of intersection of lines ab, ed, but his horizontal lines do not give the hexagons involved. In both (g) and (h) below, the horizontal lines give the hexagons involved, and, taken in pairs, the Pascal lines of these hexagons. In (g), line (2) is formed from line (1) by writing under each segment of (1) its opposite segment in (1), reversing, in each segment, the hexagon order of letters in (1); (3) from (2) as (2) from (1); lines (1), (2) indicate the Pascal of (1); (2), (3) that of (2); (3), (1) that of (3). In (h) the same procedure is followed, except that the hexagon order is retained in one column (in this particular h-point, the third) and reversed in the other two; (1), (2) give the Pascal of (1); (2), (3) that of (2); (3), (1′) that of (3); (1′) is (1).

In this notation also the quadrilaterals to which to refer the conic to prove concurrency for (g), (h) are indicated. For concurrency of (g) the Pascals may be expressed in terms of the opposite sides in any column, and the reference quadrilaterals are the lines of the two remaining columns. For concurrency of (h), we must express the Pascals in terms of the sides of the 3d column (always that column in which the hexagon order is held) and the reference quadrilaterals are the lines of the first and second columns. Note now that the third columns of (g) and (h) indi-

$$\begin{bmatrix} ab, & cd, & ef \\ ed, & af, & cb \\ cf, & eb, & ad \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \quad (g); \qquad \begin{bmatrix} ab, & cd, & ef \\ ed, & af, & bc \\ bf, & ec, & da \\ dc, & ba, & fe \end{bmatrix} \begin{matrix} (1) \\ (5) \\ (6) \\ (1') \end{matrix} \quad (h).$$

Selecting as reference quadrilaterals for (g)

$$abcd, \quad edaf, \quad cfeb,$$

the conic is

$$ab \cdot cd = ad \cdot bc \quad (1),$$

or,

$$ed \cdot af = ad \cdot ef \quad (2), \qquad (A)$$

or,

$$cf \cdot eb = ef \cdot bc \quad (3).$$

Subtracting in pairs

$$\left.\begin{aligned} ab \cdot cd - ed \cdot af &= ad\,(bc - ef), \\ ed \cdot af - cf \cdot eb &= ef\,(ad - bc), \\ cf \cdot eb - ab \cdot cd &= bc\,(ef - ad). \end{aligned}\right\} \qquad (B)$$

And the Pascals are,

$$bc - ef = 0, \qquad (4)$$

$$ad - bc = 0, \qquad (5)$$

$$ef - ad = 0. \qquad (6)$$

Identifying the conic with quadrilaterals in the first and second columns of (h), the conic is

$$ab \cdot cd = ad \cdot bc \qquad (7)$$

or

$$ed \cdot af = ad \cdot ef \qquad (8)$$

or

$$bf \cdot ec = ef \cdot bc. \qquad (9)$$

On subtracting in pairs,

$$\left.\begin{aligned} ab \cdot cd - ed \cdot af &= ad\,(bc - ef), \\ ed \cdot af - bf \cdot ec &= ef\,(ad - bc), \\ bf \cdot ec - ab \cdot cd &= bc\,(ef - ad). \end{aligned}\right\} \qquad (D)$$

And again the Pascals are,

$$bc - ef = 0, \qquad (10)$$

$$ad - bc = 0, \qquad (11)$$

$$ef - ad = 0, \qquad (12)$$

cate *the same set of opposite sides.* Thus, Steiner's theorem, as expressed in a g-point, and Kirkman's theorem, as expressed in an h-point, are capable of a single expression, namely, *if three hexagons have for pairs of opposite sides the sides of a triangle, taken in pairs, the Pascal lines of these hexagons are concurrent.* Concerning duality of properties of the complete Pascal hexagon see account of investigation by G. Veronese on pages x–xii of Vol. II, 6th ed. of Salmon-Fiedler's *Kegelschnitte.* Concerning advantages of this notation, see "Hexagon Notation" by R. D. Bohannan, in *Ohio Journal of Science* for February, 1916.

concurrent lines, with the same equations as (4), (5), (6), while (11), (12) are entirely different lines from (5), (6). (*B*) shows that (5) is the line *ed, cf; af, eb; bc, ad*; while (*D*) shows that (11), with the same equation as (5), is the line *ed, bf; af, ec; bc, ad.* And (*B*) shows that (6) is the line *cf, ab; eb, cd; ef, ad,* while (*D*) shows that (12), with the same equation as (6), is the line *bf, dc; ec, ab; ef, ad.*

The trouble is that if $ab = 0$ is the equation of *ab*, so is $k \cdot ab = 0$.

If equations in normal form (or any other specified form) are used, different multipliers are necessary to render the equations (*A*) identical; likewise for those in (*C*). If l_1, l_2, l_3 are such multipliers for (*A*), the Pascals are

$$l_1 \cdot bc - l_2 \cdot ef = 0,$$

$$l_2 \cdot ad - l_3 \cdot bc = 0,$$

$$l_3 \cdot ef - l_1 \cdot ad = 0,$$

concurrent lines.

Now a different set of multipliers would be used in (*C*) and thus different equations would result therefrom.

If $ab = 0$ is the equation in *normal form* of *ab* (and all other sides of the hexagon are assumed to be expressed in *normal form*), the equation of the Pascal line of

$$fabcde \tag{1}$$

may be written

$$f\begin{pmatrix} ad \\ ab \end{pmatrix} \cdot ab = c\begin{pmatrix} ad \\ de \end{pmatrix} \cdot de, \tag{p}$$

where $f\begin{pmatrix} ad \\ ab \end{pmatrix}$ denotes the ratio of the perpendiculars from *f* on *ad* and *ab*.

For (*p*) is evidently a line passing through the intersection of *ab, de*, a pair of opposite sides of (1). And (*p*) may also be written:

$$f\begin{pmatrix} ad \\ ab \end{pmatrix} \cdot ab - ad = c\begin{pmatrix} ad \\ de \end{pmatrix} \cdot de - ad,$$

which represents a line passing through the intersection of *af, cd*, another pair of opposite sides of (1).

(The manner of writing (*p*) from (1) is evident at a glance.)

If (1) is reversed and written

$$c\ b\ a\ f\ e\ d$$

its Pascal (in terms of *ba, ed*) is by (*p*),

$$c\begin{pmatrix} be \\ ba \end{pmatrix} \cdot ba = f\begin{pmatrix} be \\ ed \end{pmatrix} \cdot ed \tag{p'}$$

But (p') is identically the same as (*p*). For if the conic is

$$ab \cdot ed = m \cdot be \cdot da,$$

then

$$f\begin{pmatrix} ab \\ ad \end{pmatrix} = mf\begin{pmatrix} be \\ ab \end{pmatrix}$$

and

$$c\begin{pmatrix} de \\ ad \end{pmatrix} = mc\begin{pmatrix} be \\ ba \end{pmatrix}.$$

Thus the ratios of the coefficients in (p), (p') are the same.

Thus the equation of the Pascal of any hexagon may be expressed definitely in terms of any pair of opposite sides by (p). (Three equations of the same form.)

Taking the hexagons of the g-point

$$\begin{bmatrix} ab, & cd, & ef \\ ed, & af, & cb \\ cf, & eb, & ad \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \qquad (g)$$

in the forms

$$bcdefa, \qquad (1)$$

$$dafcbe, \qquad (2)$$

$$febadc, \qquad (3)$$

and applying (p) to each form, we have, as the equations of the Pascals of (1), (2), (3),

$$b\begin{pmatrix} cf \\ cd \end{pmatrix} \cdot cd = e\begin{pmatrix} cf \\ fa \end{pmatrix} \cdot fa, \qquad (\mathrm{I})$$

$$d\begin{pmatrix} ab \\ af \end{pmatrix} \cdot af = c\begin{pmatrix} ab \\ be \end{pmatrix} \cdot be, \qquad (\mathrm{II})$$

$$f\begin{pmatrix} ed \\ eb \end{pmatrix} \cdot eb = a\begin{pmatrix} ed \\ dc \end{pmatrix} \cdot dc. \qquad (\mathrm{III})$$

But (I), (II), (III) are concurrent.

$$\therefore \quad \frac{a(ed)\cdot b(dc)\cdot e(cf)\cdot d(fa)\cdot c(ab)\cdot f(be)}{a(cd)\cdot f(de)\cdot c(eb)\cdot d(ba)\cdot e(af)\cdot b(fc)} = \text{unity}, \qquad (E)$$

where $a(ed)$ denotes the perpendicular from a on ed.

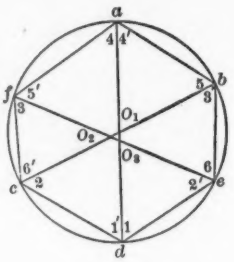
The hexagon, $abedcf$, appearing in the numerator and denominator here, is one of the three hexagons of which the selected point is the g-point. If we had expressed the Pascals in terms of the equations of pairs of opposite sides shown in any column of (g) other than the second (as above), another hexagon of which (g) is the g-point would have appeared. In the notation used in (g) the hexagons of which (g) is the g-point are shown in the columns, namely, ab, ed, cf; cd, af, eb; ef, cb, ad.

By (E) the g-point gives this theorem for any hexagon inscribed in a conic:

The product of the perpendiculars from each vertex on the second side ahead (that through the opposite vertex) in going around the hexagon in one direction, is equal to the product of the perpendiculars on the second side ahead going in the opposite direction.

And there are the usual variations for the pentagon and quadrilateral by letting two vertices of the hexagon be replaced by a tangent.

In the case of a triangle the perpendiculars going one way are the same as those going the other way.



This theorem is self-evident for a circle since it is equivalent to

$$\frac{\sin\angle 1\cdot\sin\angle 2\cdot\sin\angle 3\cdot\sin\angle 4\cdot\sin\angle 5\cdot\sin\angle 6}{\sin\angle 5'\cdot\sin\angle 4'\cdot\sin\angle 3'\cdot\sin\angle 2'\cdot\sin\angle 1'\cdot\sin\angle 6'} = \text{unity} \qquad (E') \quad \text{(see the Fig.)},$$

in which equal angles are the one under the other. Thus (E), (E') are projective relations, (E') remaining true in any conic, where the angles are not equal in pairs. The reason is that this theorem can be proved directly as a consequence of the well-known theorem, "*if from each of two points pairs of parallel secants of a conic are drawn, the ratio of the rectangle of secants from the first point is the same as that from the second*" (F). And this is a projective relation.

Draw the hexagon, *abedcf*, in this order, on a conic. Let the lines joining opposite vertices meet: *ad*, *bc* in O_1; *bc*, *ef* in O_2; *ef*, *ad* in O_3.

$$a(de) = ad\cdot\sin\angle 1,$$

$$d(fa) = ad\cdot\sin\angle 4,$$

$$b(dc) = bc\cdot\sin\angle 2,$$

$$c(ab) = bc\cdot\sin\angle 5,$$

$$e(cf) = ef\cdot\sin\angle 3,$$

$$f(be) = ef\cdot\sin\angle 6.$$

Treat the denominator in the same way and (E) will finally reduce to

$$\frac{O_1a \cdot O_1d}{O_3a \cdot O_3d} \cdot \frac{O_3e \cdot O_3f}{O_2e \cdot O_2f} \cdot \frac{O_2b \cdot O_2c}{O_1b \cdot O_1c}. \qquad (G)$$

Draw a line through O_1 parallel to $O_2O_3(ef)$, cutting the conic in A, B.

$$\therefore \quad \frac{O_1a \cdot O_1d}{O_1A \cdot O_1B} = \frac{O_3a \cdot O_3d}{O_3e \cdot O_3f}$$

and

$$\frac{O_1A \cdot O_1B}{O_1b \cdot O_1c} = \frac{O_2e \cdot O_2f}{O_2c \cdot O_2b}.$$

Eliminate $O_1A \cdot O_1B$. The result is

$$(E) = \text{unity}.$$

(G) may be written thus:

$$\frac{O_1a \cdot O_1d \cdot O_2b \cdot O_2c \cdot O_3e \cdot O_3b}{O_1b \cdot O_1c \cdot O_2e \cdot O_2f \cdot O_3a \cdot O_3d} = 1 \qquad (G')$$

and formulated into a proposition: "*if the opposite vertices of a hexagon are joined and the lines meet in three points the ratio of the product of rectangles as given in* (G') *is unity;* and corresponding propositions for the pentagon and quadrilateral.

Thus Steiner's theorem in the (g) point is a consequence of the theorem (F).

If we take the h-point

$$\begin{bmatrix} ab,\ cd,\ ef \\ ed,\ fa,\ cb \\ ca,\ be,\ fd \\ fe,\ dc,\ ba \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \\ (1') \end{matrix} \qquad (H)$$

and write the hexagons in the order

$$bcdefa \qquad (1)$$

$$dfacbe \qquad (2)$$

$$abefdc \qquad (3)$$

and apply (p) to these, the Pascals are

$$b\binom{cf}{cd} \cdot cd = e\binom{cf}{fa} \cdot fa,$$

$$d\binom{fb}{fa} \cdot fa = c\binom{fb}{be} \cdot be,$$

$$a\binom{bd}{be} \cdot be = f\binom{bd}{dc} \cdot dc,$$

and these lines are concurrent.

$$\therefore \quad \left\{ \frac{a(be) \cdot d(fa) \cdot b(cd)}{e(fa) \cdot c(be) \cdot f(dc)} \right\} \cdot \left\{ \frac{f(bd) \cdot c(fb) \cdot e(cf)}{b(cb) \cdot d(fb) \cdot a(bd)} \right\} = \text{unity}. \qquad (I)$$

The hexagon appearing here (outside the parentheses) is *adbfce*, the hexagon to which (H) is unique (none of its sides appear in (H)); *be, fa, cd* appearing in parentheses in the first fraction are diagonals joining opposite vertices. And the proposition in (I) is this:

If we go half way around a hexagon dropping perpendiculars from the vertices on the second diagonal ahead and the rest of the way dropping perpendiculars from the vertices on the side next behind; and then reverse, going in the opposite direction, the products of the perpendiculars in the two circuits are equal.

If we write the hexagons of (H)

$$edcbaf \tag{1}$$

$$cafdeb \tag{2}$$

$$febacd \tag{3}$$

and apply (p), the Pascals are

$$e\begin{pmatrix} da \\ dc \end{pmatrix} \cdot dc = b\begin{pmatrix} da \\ af \end{pmatrix} \cdot af,$$

$$c\begin{pmatrix} ae \\ af \end{pmatrix} \cdot af = d\begin{pmatrix} ae \\ eb \end{pmatrix} \cdot eb,$$

$$f\begin{pmatrix} ec \\ eb \end{pmatrix} \cdot eb = a\begin{pmatrix} ec \\ cd \end{pmatrix} \cdot cd,$$

$$\therefore \left\{ \frac{a(ce)\cdot d(ea)\cdot b(ad)}{e(da)\cdot c(ae)\cdot f(ec)} \right\} \left\{ \frac{f(eb)\cdot c(af)\cdot e(dc)}{b(af)\cdot d(eb)\cdot a(cd)} \right\} = \text{unity.} \tag{J}$$

In (J) we have, outside the parentheses, the same hexagon as in (I), *adbfce*, and the proposition of (I) is reversed:

If we go half way around a hexagon dropping perpendiculars from each vertex on the side immediately behind, and the remaining way on the second diagonal ahead; and then reverse, repeating, in the opposite direction, the products are equal.

We have previously shown that (I) and (J) are identical.

If we multiply (I) by (J), the product of the first fraction of (J) by the second of (I) is evidently unity.

$$\therefore \frac{a(be)\cdot e(cd)\cdot c(fa)\cdot f(be)\cdot b(dc)\cdot d(ab)}{a(dc)\cdot d(be)\cdot b(fa)\cdot f(cd)\cdot c(eb)\cdot e(af)} = \text{unity.} \tag{K}$$

And (K) is this:

If we go around a hexagon (here aecfbd) dropping perpendiculars on the diagonal (of opposite vertices) just ahead, and then reverse, the product of the perpendiculars in the one circuit is that of the other.

(We might have said "second diagonal ahead," and then this proposition is the same as that for a g-point, replacing the word "side" by "diagonal".)

This theorem can also be readily proven directly. If the hexagon, *adbfce*,

is drawn on a conic, and the perpendiculars expressed in terms of diagonals and sines of angles, identically the same angle relation as expressed in (E') will appear.

Thus not only are Steiner's theorem in a (g) point and Kirkman's theorem in an (h) point capable of expression in the same form, but they represent identically the same relation[1] among sines of angles between the sides and diagonals of the conic, namely (E'),—a relation of equality of sines in the case of the circle (as in (E')).

Another form of Pascal equation is, for *fabcde* (1),

$$\frac{ab}{ad}\cdot f\begin{pmatrix} ad \\ ab \end{pmatrix} + \frac{cd}{ad}\cdot e\begin{pmatrix} ad \\ cd \end{pmatrix} = 1. \qquad (q_1)$$

For (q_1) is evidently a line through the intersection of $ab = 0$ and $ad - cd\cdot e\begin{pmatrix} ad \\ cd \end{pmatrix} = 0$, or, of ab and de, a pair of opposite sides of (1). It also passes through the intersection of $cd = 0$, and $ad - ab\cdot f\begin{pmatrix} ad \\ ab \end{pmatrix} = 0$, or, of cd and af, another pair of opposite sides of (1).

The third line of (g) may be written

$$ebadcf \qquad (3)$$

and its Pascal is by (q_1),

$$\frac{ba}{bc}\cdot e\begin{pmatrix} bc \\ ba \end{pmatrix} + \frac{dc}{bc}\cdot f\begin{pmatrix} bc \\ dc \end{pmatrix} = 1. \qquad (q_2)$$

Eliminating ab between (q_1), (q_2), the coefficient of dc is

$$f\begin{pmatrix} ad \\ ab \end{pmatrix}\cdot f\begin{pmatrix} bc \\ dc \end{pmatrix} - e\begin{pmatrix} bc \\ ba \end{pmatrix}\cdot e\begin{pmatrix} ad \\ cd \end{pmatrix},$$

which is zero.

For if the conic is

$$ad\cdot bc = m\cdot ab\cdot cd,$$

$$f\begin{pmatrix} ad \\ ab \end{pmatrix}\cdot f\begin{pmatrix} bc \\ dc \end{pmatrix} = m = e\begin{pmatrix} bc \\ ba \end{pmatrix}\cdot e\begin{pmatrix} ad \\ cd \end{pmatrix}.$$

Thus the eliminant is

$$e\begin{pmatrix} ba \\ cb \end{pmatrix}\cdot bc = f\begin{pmatrix} ba \\ ad \end{pmatrix}\cdot ad,$$

which, by (p), is the Pascal of

$$ebcfad, \qquad (2)$$

the middle line of (g), which shows concurrency for (g).

By means of equations like (q_1) concurrency for (h) may be shown.

[1] See Veronese, loc. cit.

A SIMPLIFIED PROOF OF THE DISTRIBUTIVE LAW OF MULTIPLICATION GIVEN IN HILBERT'S "THE FOUNDATIONS OF GEOMETRY."

By WILLIAM E. ROTH, University of Illinois.

Professor Hilbert has developed an algebra of segments in his *Foundations of Geometry*[1] on the following assumptions:

I. The axioms of connection:

1. Two distinct points A and B always completely determine a straight line.
2. Any two distinct points of a straight line completely determine that line.

II. The axioms of order.

III. The parallel axiom.

IV. A special form of Desargue's theorem.

In the proof which follows, the axioms of order are not used and, therefore, need not be stated; the parallel axiom is that of Euclid; and, of the special form of Desargue's theorem used by Hilbert, only the part given below is applied in the demonstration:

If two triangles are so situated in a plane that the straight lines joining the homologous vertices intersect in a common point, or are parallel to one another, and, furthermore, if two pairs of homologous sides are parallel to each other, then the third sides of the two triangles are parallel to each other.

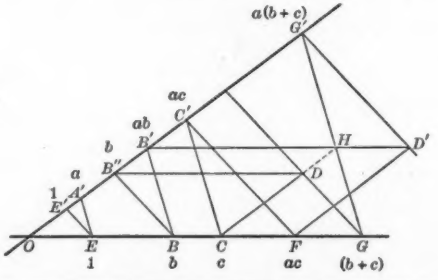
On the basis of the above assumptions, Hilbert defines both addition and multiplication of segments by constructions which will be given below. However, the associative law of multiplication is not established, and so it is desirable to establish the distributive law in two forms, namely

$$a(b + c) = ab + ac,$$

and

$$(a + b)c = ac + bc.$$

The present article aims to present for the first of these a simpler proof than that given by Dr. von Schaper in Hilbert's book.



[1] Translation by E. J. Townsend, pages 79–99.

Constructions: Take on one of two straight lines meeting in O

$$1 = OE, \quad b = OB, \quad c = OC,$$

and on the other

$$1 = OE', \quad a = OA'.$$

Draw EA' and parallel to it draw BB' and CC', which meet OE' in the points B' and C' respectively. Then by Hilbert's definition of multiplication (*l. c.*, page 81),

$$OB' = ab \text{ and } OC' = ac.$$

Draw the unit line EE' and parallel to it draw BB'' and $C'F$, which respectively meet OE' in B'' and OE in F. Then

$$OB'' = b \text{ and } OF = ac.$$

Next construct $B''D$ and $B'D'$ parallel to OE, and CD and FD' parallel to OE'; the points D and D' are uniquely determined. A straight line through D parallel to EE' cuts OE in a point G, such that by the definition of addition of segments (Hilbert, *l. c.*, page 81),

$$OG = b + c.$$

Now take GG' parallel to EA'; so again by the definition of multiplication

$$OG' = a(b + c). \tag{1}$$

If we were to draw through D' a straight line parallel to EE', it would cut OE' in a point defining the segment $ab + ac$, but instead of making this construction, we shall simply draw the straight line $D'G'$ and prove that it has the above property.

Proof: Denote the point of intersection of GG' and $B'D'$ by H. Connect D and H by a straight line. In the triangles $BB''B'$ and GDH, we have the homologous sides BB'' and BB' parallel to GD and GH respectively, and their corresponding vertices are connected by the parallel lines BG, $B''D$, and $B'H$. Then by the part of Desargue's theorem given above, DH is parallel to $B'B''$. But by construction CD is parallel to OE'; so the points C, D and H are collinear because of the parallel axiom. Therefore, in the triangles $CC'F$ and $HG'D'$, the following relations hold: CC' and CF are parallel to HG' and HD' respectively; also the lines joining the corresponding vertices are mutually parallel; consequently $G'D'$ is parallel to $C'F$ by Desargue's theorem. Then $G'D'$ is parallel to the unit line EE'. Then from the definition of addition, we have

$$OG' = ab + ac. \tag{2}$$

Then through equation (1) we get immediately

$$a(b + c) = ab + ac.$$

Thus the first form of the distributive law of multiplication is demonstrated on the basis of the given assumptions.

The above proof is simpler than that of Dr. von Schaper in that it requires only two applications of the second part of Desargue's theorem; whereas, the latter requires three applications of the first part of this theorem and five of the second part to complete it.

THE DERIVATIVE OF THE LOGARITHM.

By M. B. PORTER, University of Texas.

That the problem of deriving the logarithm presents pedagogic difficulties is sufficiently evident to any one who turns the pages of the texts on the calculus. A great many of these content themselves with showing that $[1 + (1/n)]^n$, n positive and integral, approaches $\Sigma(1/n!)$ as a limit as n becomes infinite and hence that, if $\log x \doteq \log e$ as $x \doteq e +$, the right hand incremental ratio approaches the limit $(1/x) \cdot \log_e a$ when Δx approaches zero over a certain denumerable point set.[1] Some show that this limit is the same over any point set to the right or left of x, though all assume the continuity of $\log x$. The mechanism of this proof involves the binomial theorem for positive integral exponents, simple convergence tests, and obvious inequalities. The main criticisms that can be urged against such proofs is that they are incomplete, that the binomial theorem has usually been proved by an incomplete induction, and that the proof involves many different steps. The steps are simple in themselves, but after all almost as much is assumed as is proved.

In the first edition of Vallée-Poussin's *Cours* an interesting proof of these results is obtained by means of the elementary inequality $a^{n+1} > 1+(n+1)(a-1)$, followed by the substitution of $[1 + \omega/(n + 1)] \div [1 + (\omega/n)]$ for a.[2] Here, while the steps are all elementary, the obvious artificiality of the whole process unfits it for elementary instruction; the substitution is one that the student would never invent for himself or remember. On the other hand, the Davis-Hedrick *Calculus*, frankly recognizing the unconvincing character of elaborate proof as well as its incompleteness, for the immature mind of the average beginner, makes a stronger appeal to intuition and thus obtains a greater vividness of effect by a sharper, quicker attack and produces quite as satisfactory a state of mental *bien être* on the part of the youthful and uncritical student as that obtained by the more tiresome process, thus following the safe pedagogic principle that it is not worth while to bother the student with details of proof which he cannot understand or at least whose necessity he does not appreciate.

The question of the continuity of the logarithm can be treated by constructing the values of the logarithm function by the insertion of a series of arithmetic and geometric means—the method used by Briggs in the calculation of his tables.

To many teachers of the calculus it seems desirable to put in the hands of the student a simple outline of a proof, which he can fill in, whereby the existence of

[1] Osborne's Calculus, revised ed., pp. 9–10.

[2] See Granville's *Calculus*, p. 31, where the proof is reproduced.

the limit $\lim_{n=\infty}[1+(1/n)]^n$ is demonstrated. Since the points to be established in all such proofs are the same, the only simplification possible is in the manner in which these points are established, and here it seems evident that the greatest simplification will be obtained if this process is identical for all the points involved. The writer has tried with considerable success the following procedure.

Lemma. Applying the first law of the mean twice to $(1-x)^n$ where n is rational, we have

$$\begin{aligned}(1-x)^n &= 1 - nx(1-x_1)^{n-1}\\ &= 1 - nx[1-(n-1)x_1(1-x_2)^{n-2}]\\ &= 1 - nx + n(n-1)xx_1(1-x_2)^{n-2}, \quad x > x_1 > x_2 > 0.\end{aligned}$$

Step 1°.

$$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n,$$

n rational and positive. To prove this show that

$$\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} \equiv \left(1+\frac{1}{n}\right)\left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1} \equiv \left(1+\frac{1}{n}\right)\left[1-\frac{1}{(n+1)^2}\right]^{n+1} > 1$$

by applying the lemma to the last bracket.

Step 2.° Show that $[1+(1/n)]^n$ does not increase indefinitely with n. Apply the lemma to

$$\left(1+\frac{1}{n}\right)^{-n/2}$$

and thus show that $[1+(1/n)]^n < 4$.

Step 3°. Show that

$$\lim_{n=+\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n=+\infty}\left(1-\frac{1}{n}\right)^{-n}$$

by applying the lemma to

$$\frac{\left(1+\frac{1}{n}\right)^n}{\left(1-\frac{1}{n}\right)^{-n}} \equiv \left(1-\frac{1}{n^2}\right)^n.$$

The theorem has now been proved for n rational and the remainder of the proof is filled in as usual by considering the inequality

$$\left(1+\frac{1}{n+1}\right)^n < \left(1+\frac{1}{\omega}\right)^\omega < \left(1+\frac{1}{n}\right)^{n+1}, \qquad n < \omega < n+1.$$

We can now calculate e by a table of logarithms.

A proof of this character should, of course, not be given until the use of the mean value theorem has been freely illustrated by applications to particular

functions and numerical problems. Such applications are numerous and interesting. We merely note here such as

$$\begin{aligned}
\sin x &= x - \epsilon && \text{where} && \epsilon < x^3, && \\
\cos x - 1 &= -(x^2/2) + \epsilon && \text{where} && \epsilon < x^4/4, && x < 1, \\
(1+x)^{1/n} &= 1 + (x/n) - \epsilon && \text{where} && \epsilon < x^2/n, && \\
\tan^{-1} x &= x - \epsilon && \text{where} && \epsilon < x^3, &&
\end{aligned}$$

etc., and the justification of the ordinary rules of interpolation in tables of natural sines, cosines, etc., by means of the double application of this theorem. Other applications to maxima and minima problems, and asymptotes at once suggest themselves, but it is not worth while to enter into further detail.

NEW BOOKS RECEIVED.

SOLID GEOMETRY. By William Betz and Harrison E. Webb. Ginn and Company, Boston, 1916. xxii + 178 pages. $0.75.

SOLID GEOMETRY. By John H. Williams and Kenneth P. Williams. Lyons and Carnahan, Chicago, 1916. xii + 162 pages. $0.80.

TEXT-BOOK OF MECHANICS, VOLUME VI. THERMODYNAMICS. By Louis A. Martin, Jr. John Wiley and Sons, New York, 1916. xviii + 313 pages. $1.75.

A COMMUNITY ARITHMETIC. By Brenelle Hunt. American Book Company, New York, 1916. vii + 277 pages. $0.60.

ANALYTIC GEOMETRY. By W. A. Wilson and J. I. Tracey. D. C. Heath and Company, Boston, 1915. ix + 212 pages. $1.20.

THEORY OF ERRORS AND LEAST SQUARES. A Textbook for College Students and Research Workers. By LeRoy D. Weld. The Macmillan Company, New York, 1916. xii + 190 pages. $1.25.

GOURSAT'S MATHEMATICAL ANALYSIS, VOLUME II, PART I. FUNCTIONS OF A COMPLEX VARIABLE. By E. R. Hedrick and Otto Dunkel. Ginn & Company, Boston, 1916. x + 259 pages. $2.75.

FIVE-FIGURE MATHEMATICAL TABLES. By E. Chappell. The D. Van Nostrand Company, London, 1915. xvi + 320 pages. $2.00.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Grundriss der Differential- und Integral-Rechnung. L. KIEPERT. I Teil: *Differential-Rechnung.* Zehnte Auflage des gleichnamigen Leitfadens von DR. MAX STEGEMANN. 1905. II Teil: *Integral-Rechnung.* Neunte Auflage, 1908. Helwingsche Verlagsbuchhandlung. Hannover.

This treatise is probably the most popular text on the calculus now available in Germany. The many merits of the work have been preserved in the new edi-

tion and a number of new subjects have been included. For an interesting review of the eighth edition see E. B. Van Vleck, *Bulletin of the American Mathematical Society*, 1896. The author has taken into account, in the later editions, some of the points raised by Professor Van Vleck.

E. J. W.

Leçons de Mathématiques Générales. By L. ZORETTI. xvi+753 pp. Gauthier-Villars, Paris, 1914. 20 francs.

This book is intended as a text to be used by those students in the French universities who, while not specializing in mathematics, find it necessary to study mathematics in preparing for their future careers. The book is admirably suited to its purpose, and American college teachers will find it interesting to note that, to a very considerable extent, the contents of this book coincide with what they usually present to their students in their courses in analytic geometry and calculus. There are included however a number of topics not usually treated in our American courses, and it would seem to be a question well worthy of serious thought, whether some or all of these subjects might not be as valuable to American as to French students of this class. The book is introduced by a preface written by Professor Appell, which discusses with great lucidity the pedagogic situation involved.

E. J. W.

Historical Introduction to Mathematical Literature. By G. A. MILLER. The Macmillan Company, New York. xiii+302 pp. $1.60.

This, the most recent product of Professor Miller's prolific pen, is a real innovation in mathematical literature. The plan and scope of the volume, its purposes and contents, make it differ in kind from any other book about mathematics with which the reviewer is acquainted.

As the author tells us, the book found its origin in a series of lectures which were intended to supplement the regular mathematical courses. Naturally the book itself has turned out to be something partaking of the nature of a supplement, exhibiting a certain lack of unity and a rather noticeable looseness of connection between its various parts. But each of these parts is itself well bound together, and the author expresses his views on a large number of questions in an interesting and forceful style, which frequently assumes the form of epigram.

Professor Miller feels, as many of us do, that something should be done to widen the perspective of our students of mathematics. He thinks that this can best be accomplished, by supplementing the detailed work, in problems and theorems, of the regular courses, by material of an informational and historical character. This is the need which we attempt to meet by "synoptic and inspirational courses." Professor Miller thinks that his book may serve as a basis for such courses, and also for a first course in the history of mathematics.

In regard to the history of mathematics, the author takes a rather novel and interesting point of view. He thinks that a first course in this subject should

discuss "recent mathematical events and developments" rather than the mathematics of the ancients, and he presents some strong arguments in favor of this view. In fact, it is evident that historical study of any sort is apt to degenerate into mere text-book work, unless the primary sources of information are, at least in part, available for the use of the student. And it is certainly not an easy matter to establish direct connections with the sources of ancient mathematics, on account of the linguistic difficulties involved. By concentrating attention on more recent developments, some genuine historical study becomes possible even for those who know English only. Again, the author points out how very essential such detailed historical investigations and comparisons become, if we wish to arrive at a correct interpretation of even such a simple historical statement, as "Newton discovered the binomial theorem." The greater ease with which this can be done for more recent mathematical developments, as compared with the obscurity which necessarily surrounds the origins of various ideas transmitted to us by the Ancients, makes it fairly evident that we can hardly hope to understand the latter except by means of the light which is thrown upon intellectual origins in general by studies of the former kind.

Of course there are serious difficulties in the way of carrying out a program of this kind for the history of mathematics. The mathematics of the ancients not only came first chronologically, but much of it comes first logically and heuristically, as a prerequisite for later developments. Thus, much of the ancient mathematics is simpler and more easily understood than most of the modern work based upon it. But the simplicity of the ancient mathematics, as contrasted with the modern, is very much exaggerated by our traditional methods of teaching, methods which are being modernized very slowly indeed. As a matter of fact, many notions of the so-called higher mathematics are much simpler, and much more important, than many of the things now taught in every high-school. Nothing could be more helpful in advancing mathematical education than a thoroughgoing revision of the mathematical curriculum of the secondary schools from this point of view. I believe that a combined synoptic and historical course, still largely based on the chronological order, but taking into account only the most essential developments of both ancient and modern times, would furnish the best solution of the educational problem in which Professor Miller is interested, namely, to secure an intelligent appreciation, by educated people in general, of the rôle of mathematics in human thought and in the history of civilization. I feel that such a plan would be even more satisfactory than that outlined by Miller, but nevertheless his point of view is very suggestive.

Professor Miller's book brings together much information of great value which is not easily available elsewhere. Chapter II, which discusses the various kinds of mathematical literature, and the appendix, "Lists of important works," alone are worth the price of the book. Every one who has attempted to direct advanced students, knows how difficult it is for them to find their way through the literature. I feel that, from now on, my troubles in this direction, at least, are at an end. I shall ask them to read Miller's "Introduction."

Chapter VII contains interesting sketches of twenty-five "deceased mathematicians." There are at least two serious omissions in this list, Jacobi and Riemann. Jacobi, to be sure, is mentioned several times in other parts of the book; but even in the sketch of Abel, no mention is made of Jacobi's fundamental contributions to the theory of elliptic functions. Riemann's name does not occur anywhere in the book, so far as I have been able to make out, although the theory of functions is mentioned several times, and although biographical sketches of Weierstrass and Cauchy are included among the twenty-five.

Professor Miller's book is a valuable contribution to popular mathematical literature, and should help to arouse a more general interest in mathematics. It deserves an honorable place in every mathematical library, and it should be read by all those who feel some interest in mathematics, especially if their mathematical education has been limited. They will, at least, find out that mathematics is not dead; that it is growing day by day, and far more rapidly now than in ancient times.

E. J. WILCZYNSKI.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

[Send all Communications to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

460. Proposed by J. J. GINSBURG, Student, Cooper Union, New York.

Find the value of $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+}}}}$ to infinity.

461. Proposed by E. T. BELL, Seattle, Washington.

(1) Two events have probabilities p, q, respectively. The events may be either (i) mutually independent; or (ii) mutually exclusive. Assign meanings to the symbol p^q, in terms of the two events, where p^q is written for $p \times p \times \cdots \times p$, ($q$ factors p), in cases (i), (ii), and $p \times p$ has the customary meaning (as a probability).

(2) What relations, if any, other than (i) or (ii) can exist between two events? Upon what postulates is the answer to this based?

462. Proposed by H. S. UHLER, Yale University.

Show how to transform A into S, where these symbols denote the equivalent formulæ for the general case of Calculus Problem No. 363, pages 52 and 54 in the February, 1916, MONTHLY:

$$A \equiv 4\pi R^2 - 4nR^2 \sin^{-1}\left(\frac{R\sin\frac{\pi}{n}}{\sqrt{R^2-a^2}}\right) + 2anR\sin^{-1}\left[\frac{2a\left(\tan\frac{\pi}{n}\right)\left(\sqrt{R^2-a^2\sec^2\frac{\pi}{n}}\right)}{R^2-a^2}\right],$$

$$S \equiv 4nR\left\{a\sin^{-1}\left(\tan\frac{\pi}{n}\cdot\frac{a}{\sqrt{R^2-a^2}}\right) - R\sin^{-1}\left[\tfrac{1}{2}\sin\frac{2\pi}{n}\cdot\frac{R-\sqrt{R^2-a^2\sec^2\frac{\pi}{n}}}{\sqrt{R^2-a^2}}\right]\right\}.$$

GEOMETRY.

492. Proposed by FRANK V. MORLEY, Student, Haverford College.

Let a_i $(i = 1, 2, 3, 4)$ be four points on a circle, and let the symmedian point of the triangle formed by omitting a_i be s_i. Prove that the four points s_i have the same diagonal triangle as the four points a_i.

493. Proposed by FLORENCE P. LEWIS, Goucher College, Baltimore.

Construct three circles each of which shall be tangent to the other two and to two sides of a given triangle.

494. Proposed by DAVID F. BARROW, University of Texas.

Students of geometry are very apt to assume that a theorem, true in general, will hold in all limiting cases. This trustfulness leads to frequent errors. An example is the following: Let C_1, C_2, C_3, C_4 denote four circles, and P_{ij}, P_{ij}' denote the two points in which C_i and C_j intersect. If P_{12}, P_{23}, P_{34}, P_{41} are concyclic on a circle C, then P_{12}', P_{23}', P_{34}', P_{41}' will be concyclic on a circle C'. This is still true if C is very small. Hence we might hastily conclude that: If four circles are concurrent, then their other intersections, taken in pairs in a cyclic order, are concyclic. Why is not this true?

CALCULUS.

410. Proposed by J. A. BULLARD, Worcester, Massachusetts.

(*a*) Find the area of the loop of the curve $x^{2q+1} + y^{2q+1} = (2q + 1)ax^qy^q$. (For $q = 1$, we have the folium of Calculus Problem No. 379.)

(*b*) Find the area between the curve and its asymptote.

[From Johnson's *Integral Calculus*.]

411. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

Prove that the volume bounded by the surface $f(x, y, z) = 0$ is $\frac{1}{3}\int\int\left(z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right)dxdy$ integrated over the area determined by projecting the surface on the xy-plane.

412. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

Given a triangular field of sides a, b, and c. Show how to divide the field into two equal parts by a straight fence so that the cost of the fence is the least.

MECHANICS.

328. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

Find the envelope of all possible trajectories when a particle is projected with a constant velocity v from a fixed point at a distance a from the center of attraction under the law of gravitation.

329. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A smooth circular table is surrounded by a smooth vertical rim. Show that the ball, whose coefficient of restitution is e, projected along the table from a point in the rim in a direction making an angle $\tan^{-1} e^{\frac{3}{2}}$ with the radius through the point, will return to the point of projection after three rebounds.

NUMBER THEORY.

246. Proposed by ALBERT A. BENNETT, Princeton University.

Prove that

$$\frac{1}{\sqrt{b}}\left[\left(\frac{a+\sqrt{b}}{2}\right)^n - \left(\frac{a-\sqrt{b}}{2}\right)^n\right]$$

is an integer for every positive integral value of n, whenever a is an odd integer, positive or negative, and $b \equiv 1 \pmod 4$.

247. Proposed by NORMAN ANNING, Chilliwack, B. C.

To dissect the triangle whose sides are 52, 56, 60 into three Heronian triangles by lines drawn from the vertices to a point within.

The word Heronian is used in the sense of the German Heronische (Wertheim, *Anfangsgründe d. Zahlenlehre*, p. 140) to describe a triangle whose sides and area are integral.

248. Proposed by E. T. BELL, Seattle, Washington.

If $u_{n+2} = 4u_{n+1} - u_n$, with $u_0 = 2$, $u_1 = 4$, prove that the triangle Δ_n, whose sides are $u_n - 1$, u_n, $u_n + 1$, has an integral area; also that all triangles, Δ_n, whose areas are integers, and whose sides are consecutive integers, are given by this process. Hence show that, as n increases, the area of Δ_n approximates $(\sqrt{3}/4)u_n^2$, and find the degree of approximation.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

447. Proposed by ELIJAH SWIFT, University of Vermont.

(*a*) A method is sought of forming an equation such that the first k figures of some root shall be given numbers. For example, form a cubic equation such that one of its roots is 1.918 +.

(*b*) Of all the equations suggested in (*a*), determine that one for which the sum of the absolute values of all the coefficients is least.

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

Let the degree of the desired equation be n, the nearest integer to the desired root be x_0, and the root itself be $(x_0 + \Delta x)$.

Choose any $\phi(x)$ of degree $(n-1)$, and form $y = f(x) = (x - x_0)\phi(x)$.

Compute

$$\Delta y = \Delta x \cdot f'(x_0) + \frac{\overline{\Delta x}^2}{\lfloor 2} f''(x_0) + \cdots + \frac{\overline{\Delta x}^n}{\lfloor n} \cdot f^{(n)}(x_0)$$

from

$$\Delta y = \Delta x \left[\phi(x_0) + \Delta x \cdot f'(x_0) + \cdots + \frac{\overline{\Delta x}^{n-1}}{\lfloor n-1} \cdot \phi^{(n-1)}(x_0) \right] \tag{1}$$

or from

$$\Delta y = \Delta x \cdot \phi(x_0 + \Delta x). \tag{2}$$

The more convenient of (1) and (2) should be chosen; usually (1) will be preferable. The computation should be carried to at least as many decimals as are required in the root; if it is important that the root should be greater than $(x_0 + \Delta x)$ rather than less, this is readily managed by slightly increasing or decreasing Δy according as $f'(x_0)$ is > 0 or < 0, with due regard to the fact that $\Delta y \doteq \Delta x \cdot \phi(x_0)$. The desired equation is $f(x) - \Delta y = 0$.

If $n = 3$, and $\phi(x) = \alpha_0 x^2 + \alpha_1 x + \alpha_2$,

$$\Delta y = \Delta x[\alpha_0 x_0^2 + \alpha_1 x_0 + \alpha_2 + (2\alpha_0 x_0 + \alpha_1)\Delta x + \alpha_0 \overline{\Delta x}^2].$$

To form a cubic equation having the root 1.918 +, let $\phi(x) = x^2 - 2x + 7$; then

$$f(x) = x^3 - 4x^2 + 11x - 14,$$

and

$$\Delta y = -.082[4 - 4 + 7 + (4 - 2)(-.082) + (1)(.082)^2] = -.5610 -.$$

Increase Δy by .001 to insure a root > 1.918; then $-\Delta y = .560$, and the desired equation is

$$f(x) - \Delta y \equiv x^3 - 4x^2 + 11x - 13.44 = 0,$$

which has a root 1.918 +.

(*b*) If, in the preceding discussion,

$$\phi(x) = \alpha_0 x^{n-1} + \alpha_1 x^{n-2} + \cdots + \alpha_{n-1},$$

the sum of the absolute values of all the coefficients of $f(x) - \Delta y \equiv (x - x_0)\phi(x) - \Delta y$ is

$$\Sigma = |\alpha_0| + |\alpha_1 - \alpha_0 x_0| + |\alpha_2 - \alpha_1 x_0| + \cdots + |\alpha_{n-1} - \alpha_{n-2}x_0| + |-\alpha_{n-1}x_0 - \Delta x \cdot \phi(x_0 + \Delta x)|. \quad (3)$$

The α's are arbitrary. Let $\alpha_i = \alpha_{i-1}x_0$, so that $\alpha_i = \alpha_0 x_0^i$; then

$$\Sigma = |\alpha_0| + |-\alpha_0 x_0^n - \Delta x \cdot \phi(x_0 + \Delta x)|.$$

The equation $f(x) - \Delta y = 0$ must, of course, become

$$\alpha_0[x^n - (x_0 + \Delta x)^n] = 0, \qquad \text{and} \qquad \Sigma = |\alpha_0|(1 + |x_0 + \Delta x|^n).$$

By choosing α_0 small enough, we can make Σ as small as we please.

This type of equation may not be what is desired, though it serves very well for practice in Horner's Method. In any case, the expression (3) for Σ is a useful guide to a happy medium of simplicity.

448. Proposed by W. D. CAIRNS, Oberlin College.

In the *Washington* (D. C.) *Times*, Mr. W. A. Dayton called attention some time ago to a curious repetition of digits in the decimal value of 1/115. If this decimal, which we print in the form .00869565217391304347826̇0, be divided by two, the result is .00434782608695652173913̇0, the fourteen-digit and eight-digit groups having been thus interchanged. A similar result, as he points out, is obtained if the original decimal value is divided by four. Mr. Dayton asks that this curiosity be explained.

SOLUTION BY H. S. BROWN, Hamilton College, N. Y.

The decimal point was misplaced in the original question. The separation of the decimal into two *groups* of digits has no bearing on the problem.

The following theorem, found in almost any algebra that has a chapter on Theory of Numbers, applies directly: "If $1/p$ be converted into a circulating decimal with $p - 1$ figures in its recurring period, p must be prime, and the recurring period being multiplied by 2, 3, $\cdots$, $(p - 1)$ will reproduce its own digits in the same order." (C. Smith's *Treatise on Algebra*, p. 502.) Proof: If

$$1/p = .\dot{a}_1 a_2 a_3 \cdots \dot{a}_{p-1},$$

we have

$$10 = a_1 p + r_1, \qquad 10r_1 = a_2 p + r_2, \qquad 10r_2 = a_3 p + r_3, \ \cdots.$$

Now p is prime to 10, for otherwise $1/p$ would not be reducible to a pure circulator; hence p is prime to $r_1, r_2, r_3, \cdots$. But as $r_1, r_2, r_3, \cdots$, are all different and there are $p - 1$ of them, they are the numbers 1, 2, 3, $\cdots$, $(p - 1)$, not of course in order. Also from the above, p is prime. Now since $r_1, r_2, r_3, \cdots$ include all numbers 1, 2, 3, $\cdots$ $(p - 1)$, it follows that when k/p is reduced to a decimal the recurring digits will be the same as before, beginning with that one for which the previous remainder was k.

In the above example, we have

$$\frac{1}{115} = \frac{1}{5 \times 23} = \frac{2}{10 \times 23}, \qquad \text{where } k = 2.$$

It is worth noting that if the decimal 1/23 of 22 places is separated into two equal groups and the sum of the digits of the first half added to the sum of the digits of the second half, the result will consist wholly of nines. Thus, $1/23 = .0434782608695652173913̇$.

48	= sum of the first eleven digits,	04347826086
51	= sum of the second eleven digits,	95652173913
99		99999999999

This result is general under the condition that the denominator is prime and the number of figures in the recurring period is even. We note also that twos and fives, as factors in the denominator of a fraction, have no effect on the number of recurring figures when the fraction is reduced to a decimal.

Also solved by E. E. WHITFORD, E. B. ESCOTT, ELIJAH SWIFT, NORMAN ANNING, R. D. BOHANNAN, GEORGE BLANCHARD, R. E. GAINS, and E. F. CANADAY.

449. Proposed by FRANK IRWIN, University of California.

Sum the expression

$$1+2\binom{k+1}{k}+3\binom{k+2}{k}+\cdots+(n-k+1)\binom{n}{k}.$$

Also show how to sum

$$1\cdot2+2\cdot3\binom{k+1}{k}+3\cdot4\binom{k+2}{k}+\cdots+(n-k+1)(n-k+2)\binom{n}{k},$$

$$1\cdot2\cdot3+2\cdot3\cdot4\binom{k+1}{k}+3\cdot4\cdot5\binom{k+2}{k}+\cdots+(n-k+1)(n-k+2)(n-k+3)\binom{n}{k},$$

etc., where $\binom{l}{k}$ is used to denote the coefficient of x^k in $(1+x)^l$.

SOLUTION BY A. M. KENYON, Purdue University.

The sum of the first n binomial coefficients in any column of Pascal's triangle is given by the formula,

$$(1)\qquad \sum_{i=0}^{n-1}\binom{k+i}{k}=\binom{k+n}{k+1}=\binom{k+n}{n-1}, \qquad \begin{cases} k=0,1,2,\cdots \\ n=1,2,3,\cdots \end{cases}$$

as may be verified and established by induction.

Making use of the notation,

$$x^{(n)}=\prod_{i=0}^{n-1}(x-i), \qquad x^{[n]}=\prod_{i=1}^{n}(x+i), \qquad n=1,2,3,\cdots$$

$$x^{(0)}=x^{[0]}=1,$$

we have

$$i^{(m)}\binom{k+i}{k}=i^{(m)}\binom{k+i}{i}=k^{[m]}\binom{k+i}{k+m}, \qquad m=0,1,2,\cdots$$

and this in (1) gives,

$$(2)\qquad \sum_{i=0}^{n-1} i^{(m)}\binom{k+i}{k}=k^{[m]}\binom{k+n}{k+1+m}, \qquad \begin{cases} k,m=0,1,2,\cdots \\ n=1,2,3,\cdots \end{cases}$$

From the relations[1] among the coefficients of the polynomial in x which result from expanding $x^{(n)}$ we find,

$$i^{[m]}=\sum_{j=0}^{m}\binom{m}{j}^2 \underline{|m-j}\, i^{(j)}, \qquad m=0,1,2,\cdots$$

whence on making use of (2)

$$(3)\qquad \sum_{i=0}^{n-1} i^{[m]}\binom{k+i}{k}=\sum_{i=0}^{m}\binom{m}{i}^2 \underline{|m-i}\, k^{[i]}\binom{k+n}{k+1+i}, \qquad \begin{cases} k,m=0,1,2,\cdots \\ n=1,2,3,\cdots \end{cases}$$

Since the problem proposed requires the sum of the first $n-k+1$ terms, we put $n-k+1$ for n,

$$(4)\qquad \sum_{i=0}^{n-k} i^{[m]}\binom{k+i}{k}=\sum_{i=0}^{m}\binom{m}{i}^2 \underline{|m-i}\, k^{[i]}\binom{n+1}{k+1+i}, \qquad \begin{cases} m=0,1,2,\cdots \\ n>k=0,1,2,\cdots \end{cases}$$

Setting $m=0, 1, 2, 3$, etc., in (4), we get

[1] See, "Some Properties of Binomial Coefficients," *Indiana Academy of Science*, 1914, p. 449.

$$\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1};$$

$$1\binom{k}{k}+2\binom{k+1}{k}+3\binom{k+2}{k}+\cdots+(n-k+1)\binom{n}{k}=\binom{n+1}{k+1}+(k+1)\binom{n+1}{k+2};$$

$$1\cdot 2\binom{k}{k}+2\cdot 3\binom{k+1}{k}+3\cdot 4\binom{k+2}{k}+\cdots+(n-k+1)(n-k+2)\binom{n}{k}$$
$$=2\binom{n+1}{k+1}+4k^{[1]}\binom{n+1}{k+2}+k^{[2]}\binom{n+1}{k+3};$$

and

$$1\cdot 2\cdot 3\binom{k}{k}+2\cdot 3\cdot 4\binom{k+1}{k}+3\cdot 4\cdot 5\binom{k+2}{k}+\cdots+(n-k+1)(n-k+2)(n-k+3)\binom{n}{k}$$
$$=6\binom{n+1}{k+1}+18k^{[1]}\binom{n+1}{k+2}+9k^{[2]}\binom{n+1}{k+3}+k^{[3]}\binom{n+1}{k+4}.$$

Excellent solutions were also received from E. B. ESCOTT, L. C. MATHEWSON, ELIJAH SWIFT, S. C. WITHERS, and the PROPOSER.

GEOMETRY.

A Correction.—Professor R. A. Johnson, Cleveland, Ohio, has called our attention to an error in Professor Clawson's solution of Geometry problem 467, page 50, of the February MONTHLY.

In lines four and five, Mr. Clawson "refers to two quadrilaterals, using the word in the sense of quadrangle, as being inversely congruent, the corresponding sides being equal and parallel but arranged in opposite orders." Manifestly, two such quadrilaterals cannot be drawn. Mr. Clawson admits the justice of this criticism and says that what he wished to state was the fact that one of the quadrilaterals would have to be turned through two right angles or "inverted" in order to have it similarly placed with the other. Mr. Clawson also says that what he meant by saying in line 5 that the corresponding sides of the quadrilaterals are arranged in opposite order was that the sides are opposite in direction.

Professor Johnson points out that the entire solution can be made rigorous, by deleting the last five words of line 5 and also the word "inversely" wherever it occurs. EDITORS.

474. Proposed by LAENAS G. WELD, Pullman, Ill.

Upon a fixed and constant base stands a system of co-planar triangles, for each of which the radius of the inscribed circle is to that of the circumscribed circle as 1 : 2. What is the locus of the vertices opposite to the given fixed base?

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Denote the length of the base by a, the varying sides by r and t, the base angle opposite the side t by θ, radius of incircle by i, and radius of circumcircle by R.

Now

$$R = t/2 \sin\theta \text{ and } 2i = (a + r - t)\tan\theta/2.$$

Hence, if $2i = R$, $t = 4(a + r - t)\sin^2\theta/2$, from which we obtain, $t = \dfrac{2(a+r)(1-\cos\theta)}{3-2\cos\theta}$.

Again,

$$t^2 = a^2 + r^2 - 2ar\cos\theta.$$

Hence,

$$4(a+r)^2(1-\cos\theta)^2 = (a^2+r^2-2ar\cos\theta)(3-2\cos\theta)^2,$$

or

$$(a^2+r^2)(4\cos\theta-5)+2ar(4+\cos\theta-8\cos^2\theta+4\cos^3\theta)=0,$$

or

$$(a-r)^2(4\cos\theta-5)=2ar(1-\cos\theta)(1-2\cos\theta)^2.$$

Now the left side of this equation is essentially negative or zero; and the right side is essentially positive or zero. Hence the only real points on the locus are the points for which both sides are zero. These are the points $r=a,\ \theta=0$; $r=a,\ \theta=60°$; $r=a,\ \theta=300°$.

Hence, the only triangles in the system are the two equilateral triangles on opposite sides of the given base.

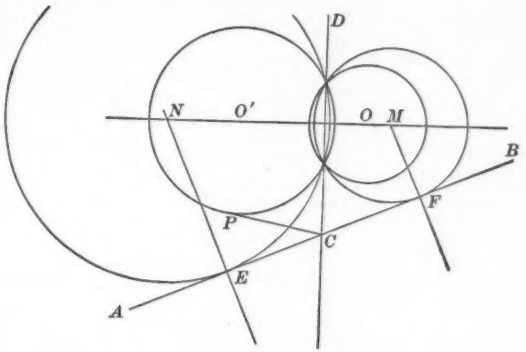
Note.—One other solution, unsigned, was received. This solution was based upon the relation between the radii of the in- and circumcircles and the distance between their centers, namely, $R^2-d^2=2Rr$; since, in the given problem, $r=\frac{1}{2}R$, $d=0$. Hence, the circles are concentric and the system of triangles reduces to two equilateral triangles on either side of the given line. EDITORS.

475. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given two circles and a straight line, to draw a circle tangent to the line and coaxial with the two given circles.

SOLUTION BY GEO. W. HARTWELL, Hamline University.

Let O and O' be the two circles and AB the given straight line. Draw the radical axis CD of the two circles. From the point of intersection C of the axis and line AB, construct a tangent to O or O'. Then with C as a center and the length of this tangent, CP, as a radius lay off a distance CE and CF on AB. E and F will be points of tangency. The points, N and M, where perpendiculars erected to AB at E and F meet the line of centers OO' will be the centers of the required circles.



Since the required circle is coaxial with O and O' its center must be on the line of centers OO'. Because the distance along a tangent drawn from any point on the radical axis to each circle of the pencil is constant E or F must be the required point of tangency. Hence, there are two solutions; one circle with center M and radius MF, the other with center N and radius NE.

Also solved by ROGER JOHNSON and H. L. AGARD.

476. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Show that the locus of the middle points of a set of parallel chords intercepted between an hyperbola and its conjugate is $4b^2x^2y^2-4a^2y^4=a^2b^4$.

SOLUTION BY J. A. CAPARO, University of Notre Dame.

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (I) and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ (II) be the equations of the hyperbola and its conjugate respectively. Let m be the slope of the set of parallel chords and $P_1(x_1y_1)$, $P_2(x_2y_2)$ be the points where the chord $y = mx + c$ intersects the primary and conjugate hyperbolas respectively; and $P(xy)$ the middle point between P_1 and P_2. Then

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \quad \text{(I)}, \qquad \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = -1 \quad \text{(II)},$$

$$y - y_1 = m(x - x_1) \quad \text{(III)}, \qquad x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2} \quad \text{(IV)}.$$

From II and IV, we have:

$$\frac{(2x - x_1)^2}{a^2} - \frac{(2y - y_1)^2}{b^2} = -1,$$

which becomes

$$\frac{x^2 - xx_1}{a^2} - \frac{y^2 - yy_1}{b^2} = -\frac{1}{2}.$$

Combining this equation with III we get,

$$x - x_1 = \frac{a^2b^2}{2(a^2my - b^2x)};$$

from I and III we get,

$$\frac{x_1^2}{a^2} - \frac{[y - m(x - x_1)]^2}{b^2} = 1.$$

Eliminating x_1 between these two equations we have

$$\frac{1}{a^2}\left[x - \frac{a^2b^2}{2(a^2my - b^2x)}\right]^2 - \frac{1}{b^2}\left[y - \frac{a^2b^2m}{2(a^2my - b^2x)}\right]^2 = 1,$$

which, by I, reduces to

$$4\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\left(\frac{amy}{b} - \frac{bx}{a}\right)^2 = (a^2m^2 - b^2),$$

which is the general equation of the desired locus. As a particular case, if the chords are perpendicular to the x axis, then writing the equation in the form,

$$4\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\left(\frac{ay}{b} - \frac{bx}{am}\right)^2 = a^2 - \frac{b^2}{m^2}$$

and making $m = \infty$, we get

$$4b^2x^2y^2 - 4a^2y^4 = a^2b^4.$$

Also solved by PAUL CAPRON.

CALCULUS.

393. Proposed by LAENAS G. WELD, Pullman, Illinois.

Find the area of the least ellipse which can be drawn upon the face of a brick wall so as to inclose four bricks.

SOLUTION BY FRANK R. MORRIS, Glendale, Calif.

Let the ellipse be represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

then each quadrant will contain one brick. Also let m be the length and n the thickness of a brick. The ellipse must pass through the point (m, n), i. e., the equation

$$\frac{m^2}{a^2} + \frac{n^2}{b^2} = 1 \text{ must be true.}$$

From this we get

$$a = \frac{bm}{\sqrt{b^2 - n^2}}.$$

The area of the ellipse, which is easily found by integration, is πab. Substituting the above value of a in this expression we have

$$\frac{\pi b^2 m}{\sqrt{b^2 - n^2}},$$

which is a function, $f(b)$, of the independent variable b. To find the minimum value of $f(b)$ find the values of b which cause the first derivative to vanish.

$$f'(b) = \pi m \frac{2b(b^2 - n^2) - b^3}{(b^2 - n^2)^{3/2}} = 0.$$

Solving this equation we find b to be 0 or $n\sqrt{2}$. The latter value is obviously the desired one. The corresponding value of a is $m\sqrt{2}$.

Hence, the area is $\pi m\sqrt{2} \cdot n\sqrt{2}$ or $2\pi mn$.

From geometrical considerations it does not seem necessary to show that $f''(b)$ is positive and that, therefore, $f(b)$ is a true minimum.

Also solved similarly by H. S. UHLER, H. C. FEEMSTER, GEORGE W. HARTWELL, and H. L. AGARD.

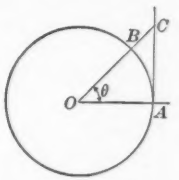
Note.—The above solution assumes that the bricks are laid side by side without mortar, whereas in a "brick wall" they are laid so as to *break joints*. Very possibly the question as answered is the one really intended by the proposer but the ellipse found certainly does not inclose four bricks as they are laid in a *brick wall*. EDITORS.

394. Proposed by W. W. BURTON, Macon, Ga.

A horse runs 10 miles per hour on a circular race-track in the center of which is an arc-light. How fast will his shadow move along a straight board fence (tangent to the track at the starting point) when he has completed one eighth of the circuit?

SOLUTION BY C. E. HORNE, Westminster College, Colorado.

Let B and C be the position of the horse and shadow at any time after starting from the point of tangency, A. Let $AB = s$, $AC = y$, angle $AOC = \theta$, and r = the radius of the ring. Then $s = r\theta$ (1) and $y = r\tan\theta$ (2). From (1), $ds/dt = r(d\theta/dt)$, the rate of the horse and from (2), $dy/dt = r\sec^2\theta(d\theta/dt)$, the rate of the shadow, $= \sec^2\theta(ds/dt) = 10\sec^2\theta = 20$ miles per hour when $\theta = \pi/4$.



Also solved by A. H. HOLMES, H. L. AGARD, J. A. CAPARO, D. RUMBLE, W. C. EELLS, CLIFFORD N. MILLS, HORACE OLSON, H. L. AGARD, and GEORGE W. HARTWELL.

395. Proposed by W. W. BURTON, Mercer University, Macon, Ga.

Into a full conical wine glass whose depth is a and whose angle at the base is 2α there is carefully dropped a spherical ball of such size as to cause the greatest overflow. Show that the radius of the ball is $a\sin\alpha/(\sin\alpha + \cos 2\alpha)$.

From Woods and Bailey's *A Course in Mathematics* (1907), Volume I, page 213.

SOLUTION BY H. S. UHLER, Yale University.

Let the sphere touch the inside of the cone near the rim with its center above the plane of the edge. Also, let h and r denote, respectively, the altitude of the submerged spherical segment and the radius of the sphere. Then $h = a + r(1 - \csc\alpha)$. The volume v of the liquid spilled is, of course, equal to the volume of the submerged spherical segment.

The volume of a spherical segment of altitude h, and radius of base r_1 is $\frac{1}{3}\pi h^3 + \frac{1}{2}\pi h r_1^2$. Here,

$$r_1 = \sqrt{2rh - h^2}, \qquad \text{hence} \qquad v = \pi h^2(r - \tfrac{1}{3}h).$$

Now,

$$\frac{dv}{dr} = \pi h\left[h + (2r - h)\frac{dh}{dr}\right], \qquad \text{and} \qquad \frac{dh}{dr} = 1 - \csc\alpha.$$

Therefore,

$$\frac{dv}{dr} = \pi h[a\csc\alpha - r(\csc\alpha - 1)(\csc\alpha + 2)].$$

Obviously, $h = 0$ cannot give a maximum overflow. Consequently,

$$r = \frac{a\csc\alpha}{(\csc\alpha - 1)(\csc\alpha + 2)} = \frac{a\sin\alpha}{(1 - \sin\alpha)(1 + 2\sin\alpha)} = a\sin\alpha/(\sin\alpha + \cos 2\alpha).$$

Furthermore,

$$\frac{d^2v}{dr^2} = -\pi h(\csc\alpha - 1)(\csc\alpha + 2).$$

Since $h \neq 0$ and $\csc\alpha > 1$, d^2v/dr^2 is negative so that the condition for a maximum is satisfied.

Also solved by A. H. WILSON, S. E. RASOR, ELMER SCHUYLER, GEORGE RAYNOR, C. N. SCHMALL, H. L. AGARD, C. N. MILLS, LEWIS CLARK, L. M. PICKETT, C. A. BERGSTRESSER, C. HORNUNG, L. G. WELD, J. A. BULLARD, G. W. HARTWELL, J. V. BALCH, J. A. WHITTED, ELIJAH SWIFT, J. C. RAYWORTH, and J. A. CAPARO.

396. Proposed by ELBERT H. CLARKE, Purdue University.

The length of the curve $y = x^n$ from the origin to the point (1, 1) is given by the formula

$$= \int_0^1 \sqrt{1 + n^2x^{2n-2}}\cdot dx.$$

Our geometric intuition would tell us that the limit of this length as n becomes infinite is 2. Give a strict analytic proof that

$$\lim_{n\to\infty} \int_0^1 \sqrt{1 + n^2x^{2n-2}}\cdot dx = 2.$$

SOLUTION BY ELIJAH SWIFT, University of Vermont.

We can easily show that

$$\lim_{n\to\infty} n^2x^{2n-2} = 0, \qquad 0 \leqq x < 1. \tag{1}$$

Next, we choose a positive number, ϵ, as small as we like. We can then find a value N of n such that for it and all larger values

$$n^2(1 - \epsilon)^{2n-2} < \eta, \tag{2}$$

where η is a positive number as small as we like. This is possible on account of (1).

Let x_n be a value of x such that $n^2{x_n}^{2n-2} = 2$.

Now write the given integral, I, as a sum of three integrals.

$$I = \int_0^{1-\epsilon} \sqrt{1 + n^2x^{2n-2}}\cdot dx + \int_{1-\epsilon}^{x_n} \sqrt{1 + n^2x^{2n-2}}\cdot dx + \int_{x_n}^1 \sqrt{1 + n^2x^{2n-2}}\cdot dx.$$

Considering values of $n > N$, we have

$$\begin{aligned}\left|\int_0^{1-\epsilon} \sqrt{1 + n^2x^{2n-2}}\cdot dx - 1\right| &= \left|\int_0^{1-\epsilon} \{\sqrt{1 + n^2x^{2n-2}} - 1\}dx - \epsilon\right| \\ &\leqq \left|\int_0^{1-\epsilon} \{\sqrt{1 + \eta} - 1\}dx\right| + \epsilon < \frac{\eta}{2} + \epsilon. \quad \text{(See (2).)}\end{aligned} \tag{3}$$

This holds if $n \geqq N$.

Now

$$\int_{1-\epsilon}^{x_n} \sqrt{1 + n^2x^{2n-2}}\cdot dx < \int_{1-\epsilon}^1 \sqrt{1 + 2}\cdot dx = \sqrt{3}\cdot\epsilon. \tag{4}$$

Finally, since for values of x greater than x_n, $n^2x^{2n-2} > 2$, we may develop $\sqrt{1 + n^2x^{2n-2}}$ into a series in descending powers of n^2x^{2n-2}, and the series will be uniformly and absolutely convergent for values of x between, and including, x_n and 1, and all values of $n \geqq N$, we have

$$\sqrt{1 + n^2x^{2n-2}} = nx^{n-1} + \frac{1}{2} \cdot \frac{1}{nx^{n-1}} - \frac{1}{2 \cdot 4} \cdot \frac{1}{n^3x^{3n-3}} + \cdots.$$

Hence,

$$\int_{x_n}^{1} \sqrt{1 + n^2x^{2n-2}} \cdot dx = 1 - x_n{}^n + \left[\frac{1}{2} \cdot \frac{1}{n(-n+2)} x^{-n+2} - + \cdots\right]_{x_n}^{1}. \tag{5}$$

Since,

$$nx_n{}^{n-1} = \sqrt{2}, \tag{6}$$

$$\lim_{n \to \infty} x_n{}^n = 0. \tag{7}$$

We may now obtain the limit of the right hand side of (5) when $n \to \infty$ by taking the limit of each term. This gives, in consequence of (6) and (7), 1 as the limit. In other words, we may choose n so large that

$$\int_{x_n}^{1} \sqrt{1 + n^2x^{2n-2}} \cdot dx = 1 + \zeta, \tag{8}$$

where ζ is a constant as small numerically as we please for all values of $n > N$.

Then

$$\int_0^1 \sqrt{1 + n^2x^{2n-2}} \cdot dx - 2 \Big| \leqq \left| \int_0^{1-\epsilon} \sqrt{1 + n^2x^{2n-2}} \cdot dx - 1 \right| + \left| \int_{1-\epsilon}^{x_n} \sqrt{1 + n^2x^{2n-2}} \cdot dx \right| + \left| \int_{x_n}^{1} \sqrt{1 + n^2x^{2n-2}} \cdot dx - 1 \right| < \frac{\eta}{2} + \epsilon + \sqrt{3} \cdot \epsilon + |\zeta|.$$

Since this may be made as small as we please by a suitable choice in order of ϵ, η, and ζ, we have proved the statement.

Also solved by TOBIAS DANTZIG.

MECHANICS.

312. Proposed by C. N. SCHMALL, New York City.

A ball of elasticity e is projected upward from a point on an inclined plane, so that after its first contact with the plane it rebounds to its starting point. If ϕ be the inclination of the plane to the horizontal, and ψ the angle made by the line of projection with the inclined plane, show that

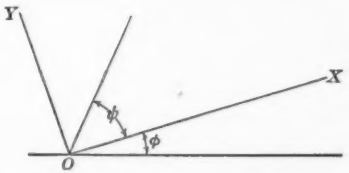
$$\cot \phi \cot \psi = e + 1.$$

I. SOLUTION BY HORACE OLSON, Chicago, Illinois.

In order that the answer given may be true the ball must be thrown in a vertical plane perpendicular to the inclined plane. Taking the axis of coordinates as indicated in the figure, the equations of the trajectory are:

$$x = vt \cos \psi - \frac{gt^2 \sin \phi}{2},$$

$$y = vt \sin \psi - \frac{gt^2 \cos \phi}{2},$$



t being the time from the instant of projection. Then,

$$\frac{dx}{dt} = v \cos \psi - gt \sin \phi \quad \text{and} \quad \frac{dy}{dt} = v \sin \psi - gt \cos \phi.$$

When the ball strikes the plane, $y = 0$,

$$t = \frac{2v \sin \psi}{g \cos \phi}, \quad \text{and } x \text{ becomes} \quad x_1 = \frac{2v^2 \sin \psi}{g \cos^2 \phi}(\cos \psi \cos \phi - \sin \psi \sin \phi),$$

whence,

$$\frac{dx_1}{dt} = v \cos x - \frac{2v \sin \psi \sin \phi}{\cos \phi}, \quad \frac{dy_1}{dt} = -v \sin \psi.$$

After the rebound,

$$x = x_1 + vt \cos \psi - \frac{2vt \sin \psi \sin \phi}{\cos \phi} - \frac{gt^2 \sin \phi}{2}, \quad y = evt \sin \psi - \frac{gt^2 \cos \phi}{2},$$

t being here reckoned from the instant of rebound. When the ball again strikes the plane, $y = 0$, and

$$t = \frac{2ev \sin \psi}{g \cos \phi} \text{ and } x \text{ becomes } x_2 = x_1 + \frac{2ev^2 \sin \psi \cos \psi}{g \cos \phi} - \frac{4e^2 v^2 \sin^2 \psi \sin \phi}{g \cos^2 \phi} - \frac{2e^2 v^2 \sin^2 \psi \sin \phi}{g \cos^2 \phi},$$

or

$$x_2 = \frac{2v^2 \sin \psi}{g \cos^2 \phi}\{(e+1)\cos \psi \cos \phi - (e^2 + 2e + 1) \sin \psi \sin \phi\}.$$

By the conditions of the problem, $x_2 = 0$, and, hence, $\cot \phi \cot \psi = e + 1$.

II. Solution by the Proposer.

Let v be the velocity of projection; then $v \sin \psi$ and $v \cos \psi$ are the components respectively perpendicular and parallel to the plane; and the components of gravity perpendicular and parallel to the plane are $g \cos \phi$ and $g \sin \phi$. Considering the perpendicular component, the time of flight is twice the time in which the velocity $v \sin \psi$ would be acquired under the action of the force $g \cos \phi$, *i. e.*, $2v \sin \psi / g \cos \phi$. After rebounding, the ball's velocity perpendicular to the plane is $ev \sin \psi$; hence, the time required in returning to the point of projection is $2ev \sin \psi / g \cos \phi$; and the whole time of flight is

$$2 \frac{v \sin \psi}{g \cos \phi}(e + 1).$$

Now, since the motion parallel to the plane is not affected by the impact, the entire time of flight is equal to twice the time in which the velocity $v \cos \psi$ would be produced by the force $g \sin \phi$, *i. e.*, $2 \frac{v \cos \psi}{g \sin \phi}$. Therefore, equating the two expressions found for the time, we have

$$2 \frac{v \cos \psi}{g \sin \phi} = 2 \frac{v \sin \psi}{g \cos \phi}(e + 1);$$

whence,

$$\cot \phi \cot \psi = e + 1.$$

NUMBER THEORY.

An excellent solution of 226 by Mrs. Elizabeth Brown Davis should have been reported in the April issue. Editors.

227. Proposed by R. P. BAKER, University of Iowa.

Show that every rational number can be expressed as a finite sum $\sum_{n=m}^{n=m+k} \frac{a_n}{n}$, where a_n is either 0 or 1 and m is any positive integer.

Solution by Frank Irwin, University of California.

Since the series $\sum_{n=m}^{n=\infty} \frac{1}{n}$ is divergent, we can find a value of r such that our given rational number, N, lies between $1/m + 1/(m+1) + \cdots + 1/(r-1)$ and $1/m + 1/(m+1) + \cdots + 1/r$ (unless indeed N is equal to an expression of this sort, in which case our problem is solved). This reduces the problem to that of expressing $N' = N - [1/m + 1/(m+1) + \cdots + 1/(r-1)]$, that is, a (proper) fraction $< 1/r$, as $\sum_{n=r}^{n=m+k} \frac{a_n}{n}$.

Let $N' = c/d$. We shall show that we may express c/d as

$$\frac{1}{l} + \frac{c_1}{d_1},$$

when l is $> r$ (else would c/d be $> 1/r$), c_1/d_1 is $< 1/l$, and $c_1 < c$. If d is divisible by c, the thing is done ($c_1 = 0$). Otherwise, let

$$d = (l-1)c + m, \quad 0 < m < c.$$

Then

$$\frac{c}{d} = \frac{1}{l} + \frac{c-m}{dl}$$

is an expression of the kind desired, since $(c-m)/dl < c/dl < 1/l$ (for c/d is a proper fraction).

Now apply the same process to c_1/d_1, which is $< 1/l$:

$$\frac{c_1}{d_1} = \frac{1}{l_1} + \frac{c_2}{d_2}, \qquad l_1 > l.$$

Continuing in this way, we shall eventually get for N' an expression of the kind we are seeking:

$$N' = \frac{c}{d} = \frac{1}{l} + \frac{1}{l_1} + \cdots + \frac{1}{l_s},$$

where

$$r < l < l_1 < \cdots < l_s;$$

for the process must come to a stop since the numerators $c_1, c_2, \cdots$ continually decrease (the process being accelerated whenever one of the fractions c_i/d_i reduces).

The procedure may be described in a few words: to keep adding to our set of fractions the largest one available (*i. e.*, one that has not already been used and whose denominator is not $< m$) that will not make their sum greater than the given rational number.

It is, perhaps, interesting to illustrate the methods employed by a numerical example. Let it be required to express 7/22 as $\sum_{n=10}^{n=10+k} \frac{a_n}{n}$.

$$\frac{7}{22} = \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{29}{660}, \qquad \text{where} \qquad \frac{29}{660} < \frac{1}{13};$$

$$\frac{29}{660} = \frac{1}{23} + \frac{7}{15180}; \qquad \frac{7}{15180} = \frac{1}{2169} + \frac{3}{32925420} = \frac{1}{2169} + \frac{1}{10975140};$$

the numerators 29, 7, 3 forming a decreasing series, and the last "remainder," 3/32925420, reducing to lower terms.

Our result then is:

$$\frac{7}{22} = \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{23} + \frac{1}{2169} + \frac{1}{10975140}.$$

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

In April, 1915, there was published in this department of the MONTHLY (Vol. XXII, pp. 119–121) a summary of remarks made by Commissioner Snedden of Massachusetts at the Cincinnati meeting of the N. E. A., Feb. 25, 1915, on the topic "Can algebra and geometry be reorganized so as to justify their retention for high-school pupils not likely to enter technical schools?" These remarks

were published to call the attention of teachers of college mathematics to a movement having for its purpose the practical elimination of algebra and geometry from the secondary-school curriculum. A question was asked in connection with the summary and the replies received were so few as to indicate that the large majority of our readers are comparatively indifferent to the movement. They may have thought that it was a matter which interested high-school teachers only.

The movement is, however, one of vital concern to teachers of college mathematics for at least two reasons. In the first place, it will not be long until the same attack will be made upon the teaching of college mathematics in other than technical schools. A revaluation and reorganization of college mathematics will be demanded, and teachers of college mathematics will do well to take some observations of the movement now in order that they may the better understand the situation when it confronts them. In the second place, if this movement is successful the teacher of college mathematics must begin his teaching on a foundation including little beyond the four fundamental operations in arithmetic.

The reader who inclines to the view that these possibilities are too remote for serious consideration will do well to read "A Modern School," by Abraham Flexner, issued by the General Education Board (61 Broadway, New York) as Occasional Paper No. 3 (23 pp.), and sent gratis on request.

The following extract from an editorial in the *New Republic* for April 8, 1916 (pp. 249–250), is doubtless somewhat typical of the way Dr. Flexner's paper will be taken by editors of other papers, especially by editors of the more radical educational journals:

> Dr. Abraham Flexner's manifesto of a modern school, issued this week by the General Education Board, cuts out at a blow all the tangled controversy of schoolmen over what shall be taught and why. The modern school would include nothing in its curriculum for which an affirmative case can not now be made out. Grammar, formal history, dead languages, formal mathematics, are in school now because of tradition and assumption. It is useless to ask whether a knowledge of these subjects is valuable. The fact is that children today do not get value from them. That is all a modern school need know. . . . Dr. Flexner's manifesto puts the whole problem as concisely as it can be put. He gives us a standard, upon which progressive educators already agree, and which can be used to test and to judge the individual school. If the teacher is to have an educational creed, this is the creed for the American school of the immediate morrow.

Dr. Flexner's guiding thesis and its bearing on mathematics are thus stated by him (p. 17):

> Modern education will include nothing simply because tradition recommends it or because its inutility has not been conclusively established. It proceeds in precisely the opposite way: *it includes nothing for which an affirmative case can not now be made out.* As has already been intimated, this method of approach would probably result in greatly reducing the time allowed to mathematics, and decidedly changing the form of what is still retained. If, for example, only so much arithmetic is taught as people actually have occasion to use, the subject will sink to modest proportions; and if this reduced amount is taught so as to serve real purposes, the teachers of science, industry and domestic economy will do much of it incidentally. The same policy may be employed in dealing with algebra and geometry. What is taught, when it is taught, and how it is taught will in that event depend altogether on what is needed, when it is needed, and the form in which it is needed.

To those of us who have been accustomed to think of mathematics as the simplest form of abstract reasoning it will seem a bit difficult to reconcile the foregoing with the following statement found on pp. 9–10:

> But in the end, if the Modern School is to be adequate to the need of modern life, this concrete training must produce sheer intellectual power. Abstract thinking has perhaps never played so important a part in life as in this materialistic and scientific world of ours—this world of railroads, automobiles, wireless telegraphy, and international relationships.

Dr. Snedden, Dr. Flexner and others who are attacking mathematics are insisting that the burden of proof to show that it should remain in the curriculum is upon those who advocate its continuance and not upon those who would eliminate it. To the writer it seems that these iconoclasts are placing entirely too much faith in the methods of experimental pedagogy so enthusiastically advocated in schools of education at the present time. Experimental pedagogy is a great improvement, but it should be borne in mind that it is only an improvement. It merely attempts to formulate explicitly experimentation which the race has long carried on informally. Humanity decided from its experience that mathematics was valuable and placed it in the course of study. It would seem that the burden of proof ought to be upon the man who would replace it with something else to demonstrate beyond question that the substitute is better than the original.

If these iconoclasts are right, the college teachers ought to be among the first to discover it and to aid in making proper readjustments. But regardless of whether they are right or wrong, teachers of college mathematics ought to be giving more thought and attention to the purposes of their teaching and the sequence and content of courses. Much remains to be done toward organizing and standardizing the undergraduate mathematical curriculum in colleges and universities. There is a somewhat general agreement as to sequence and content of courses so far as to include elementary differential and integral calculus; but beyond that point there is great variation both in opinion and practice.

It ought not to be that each college or university should be left to work out these problems by itself. There ought to be such coöperation as would enable each school to profit by the experience and counsel of others.

The editors would like to see more use made of this department in attacking these and such kindred problems as the proper differentiation of courses for groups of students specializing along different lines. Questions 2, 7, 11, 16, 27, 30 and 31 serve as a beginning in this direction. We hope that more questions will be sent in and also many more replies to questions 27, 30 and 31.

DISCUSSIONS.

Relating to "The Unification of Freshman Mathematics."

By F. L. Griffin, Reed College, Portland, Ore.

Agreeing heartily with nearly all that Mr. Nyberg says in describing his very interesting and valuable course, in the April issue of the Monthly, I reluctantly feel obliged to record a difference of opinion on certain points.

(1) *As to the comparative difficulty of his course and the one which I outlined.*

The chief difference as regards scope is that in my course elementary integral calculus replaces the more advanced parts of trigonometry, college algebra and analytics in his. It should be remembered that we carefully avoid problems in which the details are very complicated; and that we introduce the calculus gradually, returning to it again and again during the year, and thus driving it home with a minimum of difficulty; also, that elementary integral calculus is much easier for the average student to understand than the more advanced and abstract parts of trigonometric analysis, such as the general treatment of imaginaries and vectors, progressive waves, the solution of cubics, etc., or the study of higher plane curves, hyperbolic functions, etc. The fact is that my course formerly included more of these topics and less integral calculus, and was noticeably more difficult then. Consequently, I am unable to conceive how his course could succeed in any institution where mine would fail.

Indeed I do not understand why anyone should expect my course to fail with the class of students found in our state universities. After having taught the course five years to students whose preparation has varied all the way from a year of algebra and a term of geometry to four years of mathematics, I should expect the course to succeed almost anywhere with good teaching. Our students work well, but their *preparation* is in no way unusual. Their knowledge of the technique of algebra and the facts of geometry is decidedly defective.

(2) *As to the importance of including integral calculus.* It ought to be possible for prospective workers in physical chemistry, biology, statistics, or economics to get in one year the mathematical equipment which they need, including some understanding of the elementary applications of integration. If any university is not interested in providing such equipment (supposing this to be feasible), so much the worse for that university. Moreover, for physicists, integral calculus is many times as useful as differential. Again, the student specializing in mathematics should become familiar with both branches of the calculus as early as possible. Also "the non-specialist who wishes an introduction to mathematics" should by all means see something of integral calculus. The inclusion of this subject tends decidedly to unify the whole, and to give the student a "knowledge of calculus" that will really "compensate" him for the omission of other topics. Why allow the complex variable, and the witch, cissoid, conchoid, and polar equations of conics to shut out the most valuable part of the course?

(3) Finally, my statement that "the more theoretical parts of college algebra and analytics are postponed to the junior year" seems to have been misleading. The analytics as a whole is not postponed, but only the study of the more intricate parts. All the usual type equations are covered—in fact, all the plane analytics needed for the usual work in calculus.

Relating to the Slope-angle of a Curve.

By H. B. Phillips, Massachusetts Institute of Technology.

The recent review of my analytic geometry by Dr. G. R. Clements contains a rather severe criticism of my use of the angle φ from the x-axis to a line. If I

had not thought the matter over so carefully, I should heartily endorse his criticism.

The angle φ as I use it is not "the least positive angle" but any angle, positive or negative, from the positive direction of the x-axis to the line. Precisely which angle is to be represented by φ in a particular case, is determined by convenience. The tangent formulæ are correct for all.

My reason for this is to avoid the discontinuity that otherwise occurs. According to the "least positive angle" definition a line with a small positive slope makes with the x-axis a small angle while a line with a small negative slope makes an angle near 180°. This leads to trouble when an angle passes through zero. For example, the curvature is usually defined as $d\varphi/ds$. I do not know of any calculus text where this formula is considered inexact at points where $\varphi = 0$. Yet that is the case if φ is the least positive angle.

In defining angles, several cases are to be considered. In determining a directed line, angles congruent, modulo 2π, are equivalent. In determining merely the position of a line, angles congruent, modulo π, are equivalent. In three dimensions there is often no distinction between positive and negative angles. In each of these cases it is not only unnecessary but actually inconvenient to fix the angle more precisely.

OFFICIAL ANNOUNCEMENTS OF THE COUNCIL.

During the past three months, the Council has taken official action on pressing matters which seemed to require immediate action. In general, the members of the Association will be informed of pending actions in order that their opinions may reach the Council before decisions are made. These questions did not appear to be of that kind.

The actions submitted by mail and approved thus far are as follows:

(1) The Council authorized the election to membership of all those who signed the original Call for the Columbus meeting, and of those who participated in that meeting.

(2) The Council authorized the Secretary to allow a few days, not to exceed ten days, of grace for mail to reach him after April 1st, so that those desiring to enter as charter members should not be excluded by accident or by unavoidable delay.

(3) The Council ordered that the books and magazines belonging to the Association be deposited in the Oberlin College Library, under direct charge of the Secretary of the Association and the Librarian of Oberlin College. It is understood that these books are to be kept in a separate and safe place, and that they are to be subject to withdrawals and to requests from members for temporary loan, subject to the further wishes of the Association. See also (7) below.

(4) The Council ordered that a summer meeting be held near the time of the summer meeting of the American Mathematical Society, in the buildings

of the Massachusetts Institute of Technology, which institution had formally invited the Association to hold such a meeting. A Committee on the Program and a separate Committee on Arrangements were authorized; and the President was asked to appoint these committees.

(5) The Council authorized the formation of a Committee on Mathematical Requirements, to be appointed by the President.

This Committee has now been appointed as follows: J. W. YOUNG, Dartmouth College, Chairman; A. R. CRATHORNE, University of Illinois; E. H. MOORE, University of Chicago; D. E. SMITH, Columbia University; and OSWALD VEBLEN, Princeton University. It is hoped that a tentative report and announcement of plans may be ready for presentation at the coming summer meeting.

(6) The Council ordered that the Secretary-Treasurer shall pay to the treasurer of each section of the Association all money actually received by him in payment of initiation fees of new members in the territory of that section, provided, however, that this money shall be used exclusively for the payment of the incidental expenses of the section itself, and that in no case shall the fees be returned to the individual initiates.

(7) The Council authorized the appointment of a committee to be known as the Library Committee. This committee will not only have general charge of the books and magazines belonging to the Association but it will have the larger function of stimulating by every feasible means the wider knowledge and use of mathematical books. Some proposed plans have already been referred to in connection with lists of books suitable for college libraries. These and other matters will be taken up at an early date. The committee has not yet been appointed.

(8) The Council authorized the appointment of a committee to have charge of a Bureau of Information. The duties of this committee will consist in answering inquiries aside from those pertaining to the solution of problems. A more complete statement concerning the work of this committee along with the names of the committee will be announced later.

The President has appointed the following committees for the summer meeting, in accordance with the authorization of the Council. In each case, the first named member is the chairman.

Program Committee: H. B. FINE, Princeton University; DUNHAM JACKSON, Harvard University; and C. S. SLICHTER, University of Wisconsin.

Committee on Arrangements: H. W. TYLER, Massachusetts Institute of Technology; H. E. HAWKES, Columbia University; G. D. OLDS, Amherst College; CLARA E. SMITH, Wellesley College; and R. G. D. RICHARDSON, Brown University. This committee later added to its number G. W. EVANS, Charlestown High School; H. D. GAYLORD, Browne and Nichols School; W. R. RANSOM, Tufts College, and MABEL M. YOUNG, Wellesley College.

These committees are authorized to fill any vacancies which may occur and to add to their own membership other members of the Association. They

have power to make all arrangements for the summer meeting. Announcements will be issued by them in due course, in the MONTHLY, and separately through the Secretary of the Association.

The Council has decided to hold the annual meeting in connection with the meeting of the American Association for the Advancement of Science during the Christmas vacation. The meeting this year will be in New York City.

E. R. HEDRICK,
PRESIDENT OF THE ASSOCIATION.

THE SUMMER MEETING OF THE ASSOCIATION.

It is definitely determined that the first summer meeting of the Association will be held at the new buildings of the Massachusetts Institute of Technology in Cambridge, Mass., on Friday and Saturday, September first and second.

The Program Committee has been asked to decide upon the topics to be discussed in the papers to be read at the meeting and to select the members of the Association who are to be invited to prepare these papers. The Committee will be glad to receive and to consider any suggestions which members of the Association may wish to send them relative to this matter. Numerous topics and speakers are under consideration and full announcement will be made in August by letter addressed to each member of the Association. All communications in regard to the program should be addressed to Professor H. B. FINE, Princeton, New Jersey.

The Association headquarters will be at Riverbank Court, Cambridge, where rooms may be reserved in advance. The Committee on Arrangements will also have further announcements to make concerning accommodations in both Cambridge and Boston. In particular, special arrangements will be made for ladies attending the meetings and correspondence on this point may be addressed to Professor CLARA E. SMITH, or to Dr. MABEL M. YOUNG, Wellesley College, Wellesley, Mass.

A regular summer excursion railroad rate of thirty dollars for the round trip from Chicago to Boston, and correspondingly low rates from certain other points will be available. Inquiry should be made of the local ticket agents. Persons arriving in Boston by land or water and desiring to reach the Massachusetts Institute of Technology, or Riverbank Court, should inquire for trolley car for Harvard Bridge, and leave the car at the Cambridge end of the bridge. Kendall Station of the Cambridge subway is five minutes' walk from the Institute. Mail and express may be addressed, at the time of the meeting, to Riverbank Court, Cambridge, or to the Mathematical Department, Massachusetts Institute of Technology.

A dinner for members will be arranged on Friday evening, September first, at Riverbank Court, and luncheon will be served there between sessions on Friday and Saturday. If consistent with the plans of the Program Committee, there will be an automobile excursion to Wellesley, Mass., starting about four o'clock

on Saturday, with a brief visit to the college, optional dinner at the Wellesley Inn, and return by train.

Members expecting to attend the meetings will greatly assist the Committee on Arrangements by notifying the Secretary of the Committee, Mr. H. D. GAYLORD, 104 Hemenway St., Boston, Mass., (*a*) whether they desire a list of convenient boarding places; (*b*) whether they are likely to attend the Association dinner on Friday evening (probable cost $1.50); (*c*) whether they are likely to join in the Wellesley excursion and supper on Saturday (probable cost $1.50).

NOTES AND NEWS.

SEND ALL COMMUNICATIONS TO D. A. ROTHROCK, Indiana University.

Dr. GEORGE A. PFEIFFER has been appointed instructor in mathematics at Princeton University.

Professor C. J. KEYSER, of Columbia University, and Professor M. W. HASKELL, of the University of California, will exchange chairs for the first half-year 1916–17.

At the University of Oklahoma, Mr. C. T. LEVY, of the University of California, has been appointed instructor in mathematics, to take the position occupied by Mr. E. D. Meacham, who has been granted leave of absence for one year to study at Harvard University.

Mr. R. E. GILMAN, of Princeton University, has been appointed instructor in mathematics at Cornell University.

Professor A. H. NORTON, head of the department of mathematics of Elmira College, N. Y., has been appointed vice-president of that college.

Dr. GEORGE SARTON, formerly of Ghent and editor of *Isis* has been appointed lecturer on the "history of science" at Harvard University. He will give one course next year on "The origin and development of Greek science," and one on "The principles of mathematics historically considered."

Mr. ARCHIE S. MERRILL has been appointed to an assistant professorship of mathematics at the University of Montana. He is a candidate for the doctorate at the University of Chicago at the coming summer convocation.

Professor E. R. HEDRICK gave a series of three mathematical addresses at the University of Iowa on May 26 and 27, on the special invitation of those interested in forming an Iowa Section of the ASSOCIATION.

The quarter-centennial anniversary of the founding of the University of

Chicago was celebrated with due ceremony during the four days of June 3–6, 1916. Among the features of the program was a series of departmental conferences held especially in honor of those who hold the doctorate at the University, now numbering nearly one thousand. The departments of mathematics, mathematical astronomy, and physics joined together in two conferences and a dinner and social gathering. At one of these meetings there were brief reports on research activities by three doctors from the department of physics of the University of Chicago and two from the department of mathematics, namely, Professor OSWALD VEBLEN of Princeton University and Professor ARNOLD DRESDEN of the University of Wisconsin. At the other meeting there were three addresses as follows: "The Problems of Astrophysics," by GEORGE ELLERY HALE, Director of the Solar Observatory at Mount Wilson; "The Relation of Pure Science to Industrial Research," by JOHN J. CARTY, chief engineer of the American Telegraph and Telephone Company; "Current Tendencies in Mathematical Research," by EDWARD B. VAN VLECK, professor of mathematics at the University of Wisconsin. These three were among those on whom was conferred the honorary degree of Doctor of Science at the Convocation exercises on June sixth.

In 1915 a Joint Committee on Classification of Technical Literature was appointed by delegates from thirty-two technical bodies. The representative of the American Mathematical Society on this Committee is Professor E. V. HUNTINGTON, of Harvard University. The Sub-committee of the Society associated with Professor HUNTINGTON consists of Professor R. C. ARCHIBALD of Brown University, Professor T. H. GRONWALL of Princeton University, Professor E. H. MOORE of the University of Chicago, and Professor E. B. WILSON, of the Massachusetts Institute of Technology. This sub-committee is considering a suitable classification of mathematical literature.

Professor WEBSTER WELLS, of the Massachusetts Institute of Technology, died in Boston on May 23, 1916. He graduated from the Institute in 1873 and immediately joined the teaching staff. In 1893 he was made professor of mathematics, a position which he held until 1911 when he retired. He was the author of a series of textbooks in mathematics.

At Brown University three fellowships, each with a stipend of five hundred dollars, have been awarded to students who are to pursue graduate work in mathematics at the university during 1916–1917. The recipients of the awards are: BANCROFT HUNTINGTON BROWN, of Hyde Park, Mass., Grand Army Fellow; MARION ELIZABETH STARK, Norwich, Conn., Lyra Brown Nickerson Fellow; MARIAN MARSH TORREY, of Providence, R. I., Emma Josephine Arnold Fellow. Professor R. G. D. RICHARDSON has been granted leave of absence and he will probably spend the year in Cambridge, Mass.; Dr. W. BURGESS, instructor in mathematics at Cornell University and a Rhodes scholar from Brown, has been appointed instructor in mathematics; Mr. R. L. BLANCHARD has been appointed assistant in mathematics.

The National Academy of Sciences recently elected nine new members, including G. A. BLISS, professor of mathematics at the University of Chicago. Among the newly elected members of the American Philosophical Society are MAXIME BÔCHER, professor of mathematics at Harvard University, and F. R. MOULTON, professor of astronomy at the University of Chicago.

The twelfth annual session of the Association of Ohio Teachers of Mathematics and Science was held at Ohio State University on April 21 and 22. This meeting was really a joint session of the above mentioned association with the Ohio Academy of Science, the Ohio College Association and the Ohio Section of the Mathematical Association of America. General meetings of all the co-operating associations were held on Friday evening and Saturday morning; sectional meetings were held on Friday. Among the papers of interest to readers of the MONTHLY may be mentioned: "The training of science and mathematics teachers," by Professor G. R. TWISS, of Ohio State University; "Mathematics and the college curriculum," by Professor A. D. PITCHER, of Western Reserve University; "Relation of the newly organized Mathematical Association of America to the Association of Ohio Teachers of Mathematics and Science," by Professor C. C. MORRIS, of Ohio State University; "Supervised study in high school mathematics," by Mr. W. B. SKIMMING, of East High School, Columbus; "Correlated secondary mathematics," by Professor O. L. DUSTHEIMER, of Baldwin-Wallace College. A full report of the Ohio Section meeting of the ASSOCIATION is found elsewhere in this issue.

The Association of Mathematics Teachers of New Jersey held its fourth regular session at Princeton University on May 6. The program consisted of the report of the committee on "Courses in Trigonometry," by Professor C. O. GUNTHER, of Stevens Institute; also the following papers: "Euclid's theory of incommensurable magnitudes," by Professor H. B. FINE, of Princeton University; "Ptolemy's theorem," by Mr. E. FLORENCE, of Rutgers College; "An exposition of Napier's principle of logarithms," by Mr. E. S. INGRAM, of Rutgers College; "Certain religious implications of the mathematical infinite," by Rev. F. C. DOAN, of Summit, N. J.; and "The ultimate aim of a course in arithmetic," by Professor J. C. STONE, of Montclair Normal School.

The Mathematics Club of Albion College, Albion, Mich., was organized on Jan. 17, 1911, with fifteen members present. Membership is limited to those who have had at least two years of college mathematics, and who propose to continue their work along mathematical lines. Membership is gained through recommendation of the head of the department of mathematics and by vote of the club. The membership of the club shows a total enrollment of fifty since its organization in 1911. Of this number all but two have either completed their undergraduate course at Albion or are now students in the college; nine have received the M.A. degree, nine have been granted fellowships or scholarships by different universities, and one has received the Ph.D. degree. Eighteen are

teaching in the high schools of Michigan. The Albion Mathematics Club is well organized; it meets for one hour on each Tuesday evening, at which time a regular program is presented as follows: (1) Roll-call, (2) a five-minute talk on some assigned topic, (3) topic of the evening, (4) critic's report, (5) general discussion. The secretary sends a most interesting collection of topics used for response at roll-call, those used for five-minute talks, and those used as the general topic of the evening.

The following courses are announced at summer sessions:

COLUMBIA UNIVERSITY SUMMER SESSION (July 10–August 18).—By Professor M. W. HASKELL: Differential equations, five hours; Modern analytic geometry, five hours.—By Professor JAMES MACLAY: Theory of geometric constructions, five hours.—By Professor EDWARD KASNER: Theory of functions of a real variable, five hours.—By Professor W. B. FITE: Higher algebra, five hours.

CORNELL UNIVERSITY SUMMER SESSION (July 6–August 16).—By Professor VIRGIL SNYDER: Geometric constructions for high school teachers, five hours; Seminar in algebraic geometry.—By Professor W. A. HURWITZ: Mathematical analysis, five hours; Supplementary problems in algebra for high school teachers, five hours; Seminar in integral equations.—By Professor F. W. OWENS: Projective geometry, five hours; Seminar in foundations of geometry.

The Bureau of the Census, Department of Commerce, has just issued a set of "Life Tables," the first of their kind which have ever been prepared by the United States government. These tables, compiled in the division of vital statistics, under the direction of Professor J. W. GLOVER, professor of mathematics and insurance in the University of Michigan, show death rates and expectation of life at all ages for the population of the six New England states, New York, New Jersey, Indiana, Michigan, and the District of Columbia, on the basis of the population in 1910 and the mortality for 1909, 1910, and 1911. The Bulletin includes twenty-five Life Tables covering important classifications of the population, such as white, negro, native white, foreign-born white, city and urban. It is remarkable that the government has taken a census every ten years for a century, and has diligently collected mortality statistics, yet has never reduced this mass of facts to the form of Life Tables. Great credit is due Professor Glover for his labor during the past three years in planning and supervising this work. The first edition of about 25,000 copies will be ready for distribution soon, and may be had by addressing Director S. L. ROGERS, Bureau of the Census, Washington, D. C.

In the December number of *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht* is published an interesting paper by Professor W. LOREY, Leipzig, on the early life of Karl Weierstrass, the noted German mathematician, the one-hundredth anniversary of whose birth was celebrated on October 31, 1915. The paper by Lorey is devoted chiefly to the interesting events of Weierstrass's early life, some of which are here given. Weierstrass was born in the village of

Ostenfelde, on October 31, 1815, and died in Berlin, February 19, 1897. He received his early training in the gymnasium at Paderborn. His interest in mathematics was aroused while in the gymnasium by accidentally finding in the library an uncut number of *Crelle's Journal*, containing some of the beautiful geometrical work of Steiner. His interest for pure geometry, awakened by this article, remained with him throughout his mathematical career. In his Berlin lectures in later life, Weierstrass would frequently turn away from his analytical interests and give a course of lectures on some phase of pure geometry. In 1834 he left the gymnasium at Paderborn and entered the University of Bonn, not as a student of mathematics, but of law. His interest in mathematics was somewhat incidental while at the University; he studied some of the work of Laplace, but was not influenced by the mathematical lectures at Bonn. The first mathematician to make a lasting impression upon Weierstrass was Professor C. Gudermann (1798–1851) of Münster, who had already published a number of researches in spheric geometry, hyperbolic functions, and elliptic functions, in the early volumes of Crelle. Through a fellow student at Bonn, Weierstrass had come into possession of a copy of Gudermann's lectures on elliptic functions; he was so attracted by the subject that he left the University of Bonn and entered the University of Münster, 1839, to carry on work with Gudermann. He remained but one semester in Münster as a student with Gudermann, having written, however, an important memoir on "The development of modular functions," one on pedagogy, and two on philologic-historic problems. His mathematical work on "modular functions" was intended as a dissertation, but as the faculty at Münster did not grant the doctorate he did not receive that distinction until several years later, when the University of Königsberg granted him the degree. Weierstrass qualified as a gymnasium instructor in 1840, and continued in this service for sixteen years, during which time he taught a variety of subjects—science, mathematics, physical training. Remarkable mathematical investigations on algebraic functions carried on during the latter part of this gymnasium period (1848–1856), attracted nation-wide attention to him, so that in 1856 he was called to Berlin, first as instructor in a *Technical Academy* with a minor position in the University of Berlin. These positions he held until 1864, when he was promoted to a full professorship of mathematics at the University, a position in which he continued until his death in 1897.

The career of Weierstrass was very different from that of most brilliant mathematicians. The creative work of such men as Pascal, Lagrange, Abel, Galois, and many others, was done, or at least mapped out, very early in life, from 18 to 30 years of age. Weierstrass, having set out to become a lawyer, at the age of twenty-five became somewhat interested in mathematics, and then spent sixteen years in elementary teaching. He really began his remarkable mathematical investigations at the age of thirty-three, and did not enter upon service in the university until one half of his life had passed. His remarkable work was all accomplished after the age at which many brilliant mathematicians cease their research work.

The next issue of the MONTHLY will be early in September.

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H. E. SLAUGHT W. H. BUSSEY R. D. CARMICHAEL

WITH THE COÖPERATION OF

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B. F. FINKEL	L. C. KARPINSKI	G. H. LING
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